

COMPACT MODULI SPACES FOR SLOPE-SEMISTABLE SHEAVES ON HIGHER-DIMENSIONAL PROJECTIVE MANIFOLDS

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ABSTRACT. We resolve pathological wall-crossing phenomena on higher-dimensional base manifolds by considering slope-semistability with respect to movable curves instead of divisors. Moreover, motivated by these results, we construct a modular compactification of the moduli space of vector bundles on a projective threefold that are slope-semistable with respect to a curve arising as a complete intersection of two not necessarily identical very ample divisors. This moduli space is a natural higher-dimensional analogue of the Donaldson-Uhlenbeck compactification as constructed in the surface case by Joseph Le Potier and Jun Li.

CONTENTS

1. Introduction	2
2. Preliminaries	6
3. Semiampleness of determinant line bundles	9
4. A projective moduli space for slope-semistable sheaves	17
5. Geometry of the moduli space	25
6. Application to wall-crossing problems	31
References	39

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1. INTRODUCTION

Moduli spaces of semistable vector bundles and coherent sheaves on a given algebraic variety play a central role in modern day Algebraic Geometry for a number of reasons: they provide examples of higher-dimensional varieties (e.g. of hyperkähler manifolds) with a geometry that is on the one hand rich and interesting, yet on the other hand manageable. Moduli spaces of sheaves are naturally associated with the underlying variety and can therefore be used to define fine invariants of its differentiable structure (Donaldson invariants). They hence provide a point of close contact between complex and algebraic geometry on the one side and differential and symplectic geometry on the other side. Finally, moduli spaces of sheaves have numerous applications for example in enumerative geometry and mathematical physics.

In dimension greater than one, both Gieseker-semistability (which yields projective moduli spaces in arbitrary dimension) and slope-semistability (which is better behaved geometrically, e.g. with respect to tensor products and restrictions) depend on a parameter, classically the class of a line bundle in the ample cone of the underlying variety. As a consequence, with respect to all the points of view suggested above it is of great importance to understand how the moduli space of semistable sheaves changes when the semistability parameter varies.

In the case where the underlying variety is of dimension two this problem has been investigated by a number of authors and a rather complete geometric picture has emerged, which can be summarised as follows:

(i) A compact moduli space for slope-semistable sheaves also exists as a projective scheme. It is homeomorphic to the Donaldson-Uhlenbeck compactification, endowing the latter with a complex structure, and admits a natural morphism from the Gieseker compactification. This was proved by independently by Joseph Le Potier [LP92] and Jun Li [Li93].

(ii) In the ample cone of the underlying variety there exists a locally finite chamber structure given by linear rational walls, so that the notion of slope/Gieseker-semistability (and hence the moduli space) does not change within the chambers, see [Qin93].

(iii) Moreover, at least when the second Chern class of the sheaves under consideration is sufficiently big, moduli spaces corresponding to two chambers separated by a common wall are birational, and the change in geometry can be understood by studying the moduli space of sheaves that are slope-semistable with respect to the class of an ample bundle lying on the wall, see [HL95].

However, starting in dimension three several fundamental problems appear:

(i) While there are gauge-theoretic generalisations of the Donaldson-Uhlenbeck compactification to higher-dimensional varieties [Tia00], these are not known to possess a complex structure.

(ii) Adapting the notion of "wall" as in [Qin93], one immediately finds examples where these walls are not locally finite inside the ample cone.

(iii) Looking at segments between two integral ample classes in the ample cone instead, Schmitt [Sch00] gave examples of threefolds such that the point on the segment where the moduli space changes is no longer rational (as in the case of surfaces) but is a non-rational class in the ample cone.

Main results. In this paper we present and pursue a novel approach to attack and solve the above-mentioned problems. It is based on the philosophy that the natural "polarisations" to consider when defining slope-semistability on higher dimensional base manifolds are not ample divisors but rather movable curves, cf. [Miy87, CP11].

Given an n -dimensional smooth projective variety X , we consider the open set $P(X) \subset H_{\mathbb{R}}^{1,1}(X)$ of powers $[H]^{n-1}$ of real ample divisor classes $[H] \in \text{Amp}(X)$ inside the cone of movable curves. We show that $P(X)$ is open and that the natural map $\text{Amp}(X) \rightarrow P(X)$ (taking $(n-1)$ -st powers) is an isomorphism, see Proposition 6.5. Moreover, we show that $P(X)$ supports a locally finite chamber structure given by linear rational walls such that the notion of slope-(semi)stability is constant within each chamber, see Theorem 6.6. Moreover, any chamber (even if it is not open) contains products $H_1 H_2 \dots H_{n-1}$ of integral ample divisor classes, see Proposition 6.7. This explains and resolves the problem encountered by Qin, Schmitt, and others in their respective approaches to the wall-crossing problem.

By these results, we are thus led to the problem of constructing moduli spaces of torsion-free sheaves which are slope-semistable with respect to a *multipolarisation* (H_1, \dots, H_{n-1}) , where H_1, \dots, H_{n-1} are integral ample divisor classes on X . Here and in the following, we say that a torsion free sheaf E on X is *slope-stable* (resp. *slope-semistable*) *with respect to* (H_1, \dots, H_{n-1}) if for any coherent subsheaf F of E of intermediate rank we have

$$\frac{c_1(F) \cdot H_1 \cdot \dots \cdot H_{n-1}}{\text{rk}(F)} < \text{ (resp. } \leq) \frac{c_1(E) \cdot H_1 \cdot \dots \cdot H_{n-1}}{\text{rk}(E)}.$$

For reasons of clarity, in this version of our paper we concentrate on the case $n = 3$. Our main result is the following:

Main Theorem. *Let X be a smooth projective threefold, $H_1, H_2 \in \text{Pic}(X)$ two ample divisors, $c_1 \in H^2(X, \mathbb{Z})$, $c_2 \in H^4(X, \mathbb{Z})$, $c_3 \in H^6(X, \mathbb{Z})$ three classes, r a positive integral, $c \in K(X)_{\text{num}}$ a class with rank r , and Chern classes $c_j(c) = c_j$, and Λ a line bundle on X with $c_1(\Lambda) = c_1 \in H^2(X, \mathbb{Z})$. Denote by $\underline{M}^{\mu_{ss}}$ the functor that associates to each weakly normal variety S the set of isomorphism classes of S -flat families of (H_1, H_2) -semistable torsion-free coherent sheaves of class c and determinant Λ on X . Then, there exists a class $\hat{u}_2 \in K(X)_{\text{num}}$, a natural number $N \in \mathbb{N}^{>0}$, a weakly normal projective*

variety $M^{\mu ss} = M^{\mu ss}(r, c_1, c_2, c_3, \Lambda)$ with an ample line bundle $\mathcal{O}_{M^{\mu ss}}(1)$, and a natural transformation

$$\underline{M}^{\mu ss} \rightarrow \underline{Hom}(\cdot, M^{\mu ss})$$

with the following properties:

- (1) For any S -flat family \mathcal{F} of μ -semistable sheaves of class c and determinant Λ with induced classifying morphism $\Phi_{\mathcal{F}} : S \rightarrow M^{\mu ss}$, we have

$$\Phi_{\mathcal{F}}^*(\mathcal{O}_{M^{\mu ss}}(1)) = \lambda_{\mathcal{F}}(\hat{u}_2)^N,$$

where $\lambda_{\mathcal{F}}(\hat{u}_2)$ is the determinant line bundle on S induced by \mathcal{F} and \hat{u}_2 .

- (2) For any other triple $(M', \mathcal{O}_{M'}(1), N')$ consisting of a projective variety M' , an ample line bundle $\mathcal{O}_{M'}(1)$ on M' , and a natural number N' fulfilling the conditions spelled out in (1), one has $N|N'$ and there exists a uniquely determined morphism $\psi : M^{\mu ss} \rightarrow M'$ such that $\psi^*(\mathcal{O}_{M'}(1)) \cong \mathcal{O}_{M^{\mu ss}}(\frac{N'}{N})$.

The triple $(M^{\mu ss}, \mathcal{O}_{M^{\mu ss}}(1), N)$ is uniquely determined up to isomorphism by the properties (1) and (2).

Moreover, we obtain the following results concerning the geometry of $M^{\mu ss}$: Two slope-semistable sheaves F_1 and F_2 give rise to different points in the moduli space $M^{\mu ss}$ if the graded sheaf associated with Jordan-Hölder filtrations of F_1 and F_2 , respectively, or the support of naturally associated 2-codimensional cycles differ, see Theorem 5.5. As a consequence, we conclude that $M^{\mu ss}$ contains the weak normalisation of the moduli space of (isomorphism classes of) (H_1, H_2) -stable reflexive sheaves with the chosen topological invariants and determinant line bundle as a Zariski-open set. In particular, it compactifies the moduli space of (H_1, H_2) -stable vector bundles with the given invariants, thus answering in our particular setup an open question raised for example by Teleman [Tel08, Sect. 3.2]; see Remark 5.9.

Based on these results, it is natural to expect that the moduli space $M^{\mu ss}$ realises the following equivalence relation on the set of isomorphism classes of slope-semistable torsion-free sheaves: Two slope-semistable sheaves F_1 and F_2 give rise to the same point in the moduli space $M^{\mu ss}$ if and only if the graded sheaf associated with the respective Jordan-Hölder filtrations of F_1 and F_2 , as well as the naturally associated 2-codimensional cycles coincide. Comparing with the description of the geometry of the known topological compactifications of the moduli space of slope-stable vector bundles constructed by Tian [Tia00], we expect that the moduli spaces $M^{\mu ss}$ will provide new insight concerning the existence question for natural complex structures on these higher-dimensional analogues of the Donaldson-Uhlenbeck compactification.

Methods used in the proof. The proof of the main result follows ideas of Le Potier [LP92] and Jun Li [Li93] in the two-dimensional case; see also [HL10, Chap. 8] for a very nice account of these methods: first, using boundedness we parametrise slope-semistable sheaves by a locally closed subscheme $R^{\mu ss}$ of a suitable Quot-scheme. Isomorphism classes of semistable sheaves correspond to orbits of a special linear group G in $R^{\mu ss}$. We then consider a certain determinant line bundle \mathcal{L}_2 on $R^{\mu ss}$ and aim to show that it is generated by G -invariant global sections. Le Potier mentions in [LP92, first lines of Sec. 4] that in the case when $H_1 = \dots = H_{n-1} =: H$ his proof of this fact in the two-dimensional case could be extended to higher dimensions if a restriction theorem of Mehta-Ramanathan type were available for Gieseker- H -semistable sheaves. Indeed, such a result would be needed if one proceeded by restrictions to hyperplane sections on X . We avoid this Gieseker-semistability issue and instead restrict our families directly to the corresponding complete intersection curves, where slope-semistability and Gieseker-semistability coincide. The price to pay is some loss of flatness for the restricted families. In order to overcome this difficulty we pass to weak normalisations for our family bases and show that sections in powers of \mathcal{L}_2 extend continuously, and owing to weak normality hence holomorphically, over the non-flat locus. The moduli space $M^{\mu ss}$ then arises as the Proj-scheme of a ring of G -invariant sections of powers of \mathcal{L}_2 over the weak normalisation of $R^{\mu ss}$. The universal properties are then established using the G -equivariant geometry of $R^{\mu ss}$.

This construction works for base manifolds of any dimension $n \geq 3$ and will be explicitly carried out in a future version of this paper.

Outline of the paper. Section 2 contains definitions and basic properties concerning determinant line bundles and semistability with respect to movable curve classes, followed by a discussion of the properties of weakly normal spaces. In Section 3.1 we discuss the restriction of flat families of semistable sheaves to complete intersection curves. The corresponding class computations in the respective Grothendieck groups are carried out in Section 3.2, and the crucial semiamplessness result for equivariant determinant bundles on Quot-schemes is proven in Section 3.3. In Section 4 the moduli space for slope-semistable sheaves is defined and its functorial properties are established, followed in Section 5 by a discussion of its basic geometry, in particular concerning the separation properties of classifying maps, and its relation to the moduli space of slope-stable reflexive sheaves. Coming back to the motivating question, the final Section 6 discusses wall-crossing in light of the newly constructed moduli spaces.

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2. PRELIMINARIES

We work over the field of complex numbers. A separated reduced scheme of finite type over \mathbb{C} will be called an *algebraic variety*. We emphasise that we do not assume varieties to be irreducible. An irreducible smooth projective variety will be called *projective manifold*, and a projective manifold of dimension three will be called *projective threefold*. All complex spaces are assumed to have countable topology.

2.1. Grothendieck groups and determinants. We start by collecting a few relevant definitions. We follow the notations of [HL10]. Let X be a smooth projective irreducible variety of dimension n . The Grothendieck group $K(X) = K_0(X) = K^0(X)$ of coherent sheaves on X becomes a commutative ring with $1 = [\mathcal{O}_X]$ by putting

$$[F_1] \cdot [F_2] := [F_1 \otimes F_2]$$

for locally free sheaves F_1 and F_2 . Two classes u and u' in $K(X)$ will be called *numerically equivalent*, denoted $u \equiv u'$, if their difference is contained in the radical of the quadratic form

$$(a, b) \mapsto \chi(a \cdot b).$$

We set $K(X)_{\text{num}} := K(X) / \equiv$. For any Noetherian scheme Z , we let $K^0(Z)$ and $K_0(Z)$ be the abelian groups generated by locally free sheaves and coherent \mathcal{O}_Z -modules, respectively, with relations generated by short exact sequences. A projective morphism $f: Y \rightarrow Z$ induces a homomorphism $f_!: K_0(Y) \rightarrow K_0(Z)$ defined by

$$f_![F] := \sum_{\nu \geq 0} [R^\nu f_* F].$$

Any flat family \mathcal{E} of coherent sheaves on a projective manifold X parametrised by a Noetherian scheme S defines an element $[\mathcal{E}] \in K^0(S \times X)$, and as the projection $p: S \times X \rightarrow S$ is a smooth morphism, we have a well-defined homomorphism $p_!: K^0(S \times X) \rightarrow K^0(S)$, cf. [HL10, Cor. 2.1.11]. Let $q: S \times X \rightarrow X$ denote the second projection.

Definition 2.1. We define $\lambda_{\mathcal{E}}: K(X) \rightarrow \text{Pic}(S)$ to be the composition of the following homomorphisms:

$$K(X) \xrightarrow{q^*} K^0(S \times X) \xrightarrow{[\mathcal{E}]} K^0(S \times X) \xrightarrow{p_!} K^0(S) \xrightarrow{\det} \text{Pic}(S).$$

We refer the reader to [HL10, Sect. 8.1 and 2.1] for more details and for basic properties of this construction.

2.2. Semistability. Let X be a projective manifold of dimension n . Semistability of torsion-free sheaves on X is classically defined with respect to a polarisation, which is an ample class H in the algebraic geometric context or a Kähler class ϕ in the complex case. Whereas for Gieseker stability the class H

is needed as such, only its $(n - 1)$ -st power appears in the definition of slope-semistability. For the latter it is therefore reasonable to consider classes of curves rather than of divisors as polarisations. This has been done in [Miy87], and has later been extended to include a discussion of semistability with respect to movable curve classes [CP11]. In case such a class is in the interior of the cone of movable curves, it was shown in [Tom10] that the corresponding degree function is given by a class of a closed positive $(n - 1, n - 1)$ -form on X . We will give here the general definition before we specialise to the case of complete intersection classes, which is central for this paper.

Definition 2.2. A curve $C \subset X$ is called *movable* if there exists an irreducible algebraic family of curves containing C as a reduced member and dominating X . A class α in the space $N_1 = N_1(X)_{\mathbb{R}}$ of 1-cycles on X modulo numerical equivalence is called movable if it lies in the closure of the convex cone generated in N_1 by classes of movable curves.

Definition 2.3. Let $[C] \in N_1$ be a movable class. Then, a coherent sheaf E on X is called *(semi-)stable with respect to $[C]$* or simply *$[C]$ -(semi)stable* if it is torsion-free and if additionally for any proper non-trivial coherent subsheaf F of E we have

$$\mu_{[C]}(F) := \frac{[\det F] \cdot [C]}{\mathrm{rk}(F)} < (\leq) \frac{[\det E] \cdot [C]}{\mathrm{rk}(E)} = \mu_{[C]}(E).$$

By replacing in the above inequality the class $[C]$ by the class $[\omega] \in H^{n-1, n-1}(X)$ of a non-zero positive form on X we obtain the notion of $[\omega]$ -(semi)stability. When H is an ample class or when ϕ is a Kähler class on X we will speak of H -semistability or of ϕ -semistability meaning stability with respect to H^{n-1} or to ϕ^{n-1} , respectively. A system (H_1, \dots, H_{n-1}) of $n - 1$ integral ample classes on X will be called a *multipolarisation*. We will call a coherent sheaf (H_1, \dots, H_{n-1}) -(semi)stable if it is (semi)stable with respect to the complete intersection class $H_1 H_2 \dots H_{n-1} \in N_1$. Once a (multi)polarisation has been fixed, we will just speak of μ -semistability or slope-semistability. A μ -semistable sheaf will be called μ -polystable if it is a direct sum of μ -stable sheaves.

Note that the notion of slope-semistability does not change when we multiply a given movable class $[C] \in N_1$ by a constant $t \in \mathbb{Q}^{>0}$.

Basic properties of semistable sheaves with respect to multipolarisations can be found in [Miy87]. We will need two properties in our particular case:

- (1) (Boundedness) The set of coherent sheaves with fixed Chern classes on X which are semistable with respect to a fixed multipolarisation is bounded. This may be proven exactly as the corresponding statement in the case of a single polarisation [Lan04, Thm. 4.2]; see also our Proposition 6.3.

- (2) (Semistable Restriction) If the coherent sheaf E is (semi)stable with respect to the multipolarisation (H_1, \dots, H_{n-1}) then there is a positive threshold $k_0 \in \mathbb{N}^{>0}$ depending only on the topological type of E such that for any $k \geq k_0$ and any smooth divisor $D \in |kH_1|$ with $E|_D$ torsion-free one has that $E|_D$ is (semi)stable with respect to $(H_2|_D, \dots, H_{n-1}|_D)$, see [Miy87, Thm. 2.5] or [Lan04, Thm. 5.2 and Cor. 5.4].

2.3. Extension of sections on weakly normal spaces. We quickly recall some notions motivated by the first Riemann extension theorem for open subsets of \mathbb{C}^N . For more information and proofs of the fundamental classical results the reader is referred for example to [AN67, §1.1] or [Fis76, Appendix to Chap. 2].

Definition 2.4. Let X be a reduced complex space, and $U \subset X$ an open subset. A continuous function $f: U \rightarrow \mathbb{C}$ is called *c-holomorphic* if its restriction $f|_{U_{\text{reg}}}$ to the regular part of U is holomorphic. This defines a sheaf $\widehat{\mathcal{O}}_X$ of c-holomorphic functions. A reduced complex space is called *weakly normal* if $\widehat{\mathcal{O}}_X = \mathcal{O}_X$.

Definition 2.5. A variety X is *weakly normal* if and only if the associated (reduced) complex space X^{an} is weakly normal in the sense of Definition 2.4 above.

Note that by [LV81, Prop. 2.24] and [GT80, Cor. 6.13], a variety is weakly normal in the sense of Definition 2.5 above if and only if it is weakly normal in the sense of [LV81, Def. 2.4] if and only if it is seminormal in the sense of [GT80, Def. 1.2].

Proposition and Notation 2.6. *For any algebraic variety (or reduced complex space) X there exists a weak normalisation; i.e., a weakly normal algebraic variety (or complex space, respectively) X^{wn} together with a finite, surjective map $\eta: X^{wn} \rightarrow X$ enjoying the following universal property: if Y is any weakly normal complex space together with a holomorphic map $\psi: Y \rightarrow X$, there exists a uniquely determined holomorphic map $\hat{\psi}: Y \rightarrow X^{wn}$ making the following diagram commutative:*

$$\begin{array}{ccc} & & X^{wn} \\ & \nearrow \hat{\psi} & \downarrow \eta \\ Y & \xrightarrow{\psi} & X. \end{array}$$

If X is any separated scheme of finite type (or complex space), then by slight abuse of notation the reduced weakly normal scheme (or complex space, respectively) $(X_{\text{red}})^{wn}$ will also be denoted by X^{wn} .

The following elementary extension result will play a crucial role in the construction of the moduli space $M^{\mu ss}$ in Section 3.3.

Lemma 2.7. *Let S be a weakly normal variety and \mathcal{L} a line bundle on S . Suppose that given any finite set S_1, \dots, S_m of closed irreducible subvarieties of S there exists a Zariski-closed subset T of S with the following properties:*

- (i) *For every $i \in \{1, \dots, m\}$ the intersection $T \cap S_i$ has codimension at least 2 in S_i , and*
- (ii) *the restriction $\mathcal{L}|_{S \setminus T}$ is semiample.*

Then, \mathcal{L} is semiample.

Proof. Let s_0 be any closed point on S . We set $S'_1 := S$, $S'_2 := \text{Sing}(S'_1)$, $S'_3 := \text{Sing}(S'_2)$, and so forth, we subsequently take S_1, \dots, S_{m-1} to be the irreducible components of the S'_i -s, and finally set $S_m = \{s_0\}$. Consider the set T provided by the assumptions of the lemma. For later use we note that s_0 is not contained in T . Then, there exists a natural number $m = m(s_0)$ and some section $\sigma \in H^0(S \setminus T, \mathcal{L}^m)$ that does not vanish at s_0 . We shall show that σ extends over T yielding an element in $H^0(S, \mathcal{L})$ that does not vanish at s_0 . Semiampness then follows by Noetherian induction.

In order to establish the desired extension, we use the stratification $S''_i := S'_i \setminus S'_{i+1}$ of S and show inductively that σ may be holomorphically extended over each point of $T \cap S''_i$. Global *regular* extension then follows from the algebraicity of the original section σ .

The assertion is true for $i = 1$, since S''_1 is regular, and hence in particular normal. Let now $i > 1$ and take any point $s \in S''_i \cap T$. Assume that σ has already been extended to $S \setminus S'_i$. Since S''_i is regular, the restriction of σ to $S''_i \setminus T$ admits an extension to S''_i . Together with the extension of σ to $S \setminus S'_i$ this yields a c -holomorphic section in a neighbourhood of s in S , extending σ . Since X is assumed to be weakly normal, c -holomorphic functions are holomorphic, so σ extends to give a holomorphic section in a neighbourhood of s . \square

3. SEMIAMPLENESS OF DETERMINANT LINE BUNDLES

3.1. Flat restriction to curves. In the proof of our central semiampness result, Theorem 3.5, sections will be produced by restricting flat families to complete intersection curves and by a lifting procedure. In this approach, we have to deal with potential non-flatness of the restricted families. The following technical result addresses this issue.

Lemma 3.1. *Let X be a projective threefold, H_1, H_2 two very ample polarisations on X , S an algebraic variety, S_1, \dots, S_m closed irreducible subvarieties of S , and \mathcal{F} an S -flat family of torsion-free sheaves on X . Then, for a general complete intersection curve X'' of elements in $|H_1|$ and $|H_2|$ there exists a closed subvariety T of S intersecting each S_i in codimension*

at least 2 such that the restriction $\mathcal{F}|_{S \times X''}$ is flat over $S \setminus T$. Moreover, denoting by X_1 the chosen element of $|H_1|$ and setting $\mathcal{F}' := \mathcal{F}|_{(S \setminus T) \times X_1}$, $\mathcal{F}'' := \mathcal{F}'|_{(S \setminus T) \times X''} = \mathcal{F}|_{(S \setminus T) \times X''}$, the following sequences of flat sheaves over $S \setminus T$ are exact:

$$(3.1) \quad 0 \rightarrow \mathcal{F}(-X_1) \rightarrow \mathcal{F} \rightarrow \mathcal{F}' \rightarrow 0, \text{ and}$$

$$(3.2) \quad 0 \rightarrow \mathcal{F}'(-X'') \rightarrow \mathcal{F}' \rightarrow \mathcal{F}'' \rightarrow 0.$$

Remark 3.2. The assertion of Lemma 3.1 should be compared with the assumptions of Lemma 2.7, as well as with the assumptions of Proposition 3.3.

Proof of Lemma 3.1. The proof of the first part is carried out in two steps.

Step 1: Restricting to a member of $|H_1|$ and then to a member of $|H_2|$: We may assume that all the irreducible components of S appear among the subvarieties S_i . For each $i \in \{1, \dots, m\}$ choose a general point $s_i \in S_i$. By Lemma 1.1.12 and Corollary 1.1.14 of [HL10], for any general hypersurface X_1 in $|H_1|$ the restriction $\mathcal{F}_{s_i}|_{X_1}$ remains torsion-free on X_1 for all $i = 1, \dots, m$. This property remains true for any $\mathcal{F}_s|_{X_1}$ where s lies in an open neighbourhood U_1 of $\{s_1, \dots, s_m\}$ in S . Since each member \mathcal{F}_s of the family \mathcal{F} is torsion-free, by tensoring the structure sheaf sequence for $S \times X_1 \subset S \times X$ with \mathcal{F} and using [HL10, Lemma 2.1.4] we see that the family $\mathcal{F}' := \mathcal{F}|_{S \times X_1}$ is still flat over S .

Next, let X_2 be any general member of $|H_2|$ and denote by X'' the (complete) intersection of X_1 and X_2 . We apply [HL10, Lemma 2.1.4] again to see that the further restriction of \mathcal{F}' to X'' is flat over U_1 .

Step 2: Restricting to a member of $|H_2|$ and then to a member of $|H_1|$: If U_1 contains all the S_i -s we are done. If not, we choose points p_1, \dots, p_N on the irreducible components of the sets $S_i \setminus U_1$ and note that for any general $X_2 \in |H_2|$ the restriction of the family \mathcal{F} to X_2 remains torsion-free in all the chosen points p_α , $\alpha \in \{1, 2, \dots, N\}$. We obtain as before an open neighbourhood U_2 of $\{p_\alpha\}_{\alpha \in \{1, 2, \dots, N\}}$ in S such that for each $s \in U_2$ the restriction of \mathcal{F}_s to X_2 remain torsion-free. Thus, by restricting first to X_2 and then to X'' we obtain flatness over U_2 as well.

Since the complement of $U_1 \cup U_2$ has codimension at least two in each S_i by construction, setting $T := S \setminus (U_1 \cup U_2)$ the flatness of \mathcal{F}'' over $S \setminus T = U_1 \cup U_2$ is already checked, and the first part of our lemma is proved.

Second part of claim: The exactness of sequence (3.1) is clear since the sheaves \mathcal{F}_s are torsion-free. By the same reasoning sequence (3.2) is exact over U_1 . Additionally, over the whole of S we have an exact sequence

$$0 \rightarrow \mathcal{T}or_1^{S \times X}(\mathcal{F}, \mathcal{O}_{S \times X''}) \rightarrow \mathcal{F}'(-X'') \rightarrow \mathcal{F}' \rightarrow \mathcal{F}'' \rightarrow 0,$$

which we decompose into two short exact sequences

$$(3.3) \quad 0 \rightarrow \mathcal{T}or_1^{S \times X}(\mathcal{F}, \mathcal{O}_{S \times X''}) \rightarrow \mathcal{F}'(-X'') \rightarrow \mathcal{E} \rightarrow 0, \quad \text{and}$$

$$(3.4) \quad 0 \rightarrow \mathcal{E} \rightarrow \mathcal{F}' \rightarrow \mathcal{F}'' \rightarrow 0.$$

We claim that $\mathcal{T}or_1^{S \times X}(\mathcal{F}, \mathcal{O}_{S \times X''})$ vanishes over $U_1 \cup U_2$. Indeed, in view of the exact sequence (3.4), as both \mathcal{F}'' and \mathcal{F}' are flat over $U_1 \cup U_2$, the same will be true also for \mathcal{E} , cf. [AM69, Chap. 2, Ex. 25]. This in turn together with flatness of $\mathcal{F}'(-X'')$ and with the exact sequence (3.3) implies flatness of $\mathcal{T}or_1^{S \times X}(\mathcal{F}, \mathcal{O}_{S \times X''})$ over $U_1 \cup U_2$. This last sheaf is zero over $U_1 \cap U_2$ and thus it will also be zero over $U_1 \cup U_2$. This completes the proof. \square

3.2. Class computations. Here we extend some class computations from the surface case [HL10, Sect. 8.2] to case of threefolds. We remark that this generalisation will work for multi-polarisations of type (H_1, H_2) where the ample divisors H_1 and H_2 may differ.

3.2.1. Setup and notation. Let X be a projective threefold, $H_1, H_2 \in \text{Pic}(X)$ two ample divisors, $c_1 \in H^2(X, \mathbb{Z})$, $c_2 \in H^4(X, \mathbb{Z})$, $c_3 \in H^6(X, \mathbb{Z})$ three classes, $r \in \mathbb{Z}_{>0}$ a positive integer, $c \in K(X)_{\text{num}}$ a class with rank r and Chern classes $c_j(c) = c_j$, and $\Lambda \in \text{Pic}(X)$ a line bundle with $c_1(\Lambda) = c_1 \in H^2(X, \mathbb{Z})$. In the following, by μ -semistable we will always mean semistable with respect to the multipolarisation (H_1, H_2) , cf. the discussion in Section 2.2.

Let $a_1, a_2 \in \mathbb{N}$ be such that $a_j H_j$ is very ample, $j = 1, 2$, let $X_j \in |a_j H_j|$ be general elements, and set $X' := X_1$, and $X'' := X_1 \cap X_2$. We denote by h_i the class of \mathcal{O}_{H_i} in $K(X)$. Choose a fixed base point $x \in X''$, and set

$$\begin{aligned} u_0(c|_{X''}) &:= -r[\mathcal{O}_{X''}] + \chi(c|_{X''})[\mathcal{O}_x] \in K(X''), \\ u_1(c|_{X'}) &:= -r h_2|_{X'} + \chi(c|_{X'} \cdot h_2|_{X'})[\mathcal{O}_x] \in K(X'), \\ \hat{u}_2(c) &:= -r h_1 \cdot h_2 + \chi(c \cdot h_1 \cdot h_2)[\mathcal{O}_x] \in K(X), \end{aligned}$$

and finally

$$w_2(c) := -\chi(c \cdot h_1 \cdot h_2 \cdot [\mathcal{O}_{X'}])h_2 + \chi(c \cdot h_2 \cdot [\mathcal{O}_{X'}])h_1 \cdot h_2 \in K(X).$$

We will denote the degree of X with respect to H_i, H_j, H_k by $d_{i,j,k} := H_i H_j H_k$.

3.2.2. Determinant line bundles of restricted families. The following result compares determinant line bundles of flat families of sheaves on X with determinant line bundles of flat families of restricted sheaves on X'' . It generalises the computations in the surface case, see [HL10, p. 223]. We will use the notation introduced in the previous paragraphs as well as the terminology of Section 2.1:

Proposition 3.3. *Let S be an algebraic variety. Let \mathcal{F} be an S -flat family of torsion-free sheaves on X with class c and fixed determinant bundle Λ such that*

- (i) the restriction $\mathcal{F}' := \mathcal{F}|_{S \times X'}$ remains flat over S ,
- (ii) the sequence

$$0 \rightarrow \mathcal{F}(-X_1) \rightarrow \mathcal{F} \rightarrow \mathcal{F}' \rightarrow 0$$

is exact,

- (iii) the further restriction $\mathcal{F}'' := \mathcal{F}'|_{S \times X''} = \mathcal{F}|_{S \times X''}$ to $S \times X''$ is flat over S , and
- (iv) the corresponding sequence

$$0 \rightarrow \mathcal{F}'(-X'') \rightarrow \mathcal{F}' \rightarrow \mathcal{F}'' \rightarrow 0.$$

is exact.

Then, there exists an isomorphism of line bundles

$$(3.5) \quad (\lambda_{\mathcal{F}''}(u_0(c|_{X''})))^{a_1 a_2 d_{1,1,2} d_{1,2,2}} \cong (\lambda_{\mathcal{F}}(\widehat{u}_2(c)))^{a_1^2 a_2^2 d_{1,1,2} d_{1,2,2}}.$$

Proof. The proof proceeds in two steps:

Step 1: The first restriction: We first execute two class computations in the Grothendieck groups $K(X')$ and $K(X)$; these will be used to compare the line bundles $\lambda_{\mathcal{F}'}(u_1(c|_{X'}))$ and $\lambda_{\mathcal{F}}(\widehat{u}_2(c))$ with the help of the auxiliary class $w_2(c)$.

On the one hand we may write

$$(3.6) \quad \begin{aligned} w_2(c)|_{X'} &= -\chi(c|_{X'} \cdot (h_1 \cdot h_2)|_{X'}) h_2|_{X'} + \chi(c|_{X'} \cdot h_2|_{X'}) (h_1 \cdot h_2)|_{X'} \\ &\equiv a_1 d_{1,1,2} (-r h_2|_{X'} + \chi(c|_{X'} \cdot h_2|_{X'}) [\mathcal{O}_x]) \\ &= a_1 d_{1,1,2} u_1(c|_{X'}) \in K(X'). \end{aligned}$$

On the other hand, owing to

$$[\mathcal{O}_{X'}] = a_1 h_1 - \binom{a_1}{2} h_1^2 + \binom{a_1}{3} h_1^3 \in K(X),$$

we obtain

$$(3.7) \quad \begin{aligned} &w_2(c) - w_2(c) \cdot [\mathcal{O}(-X')] \\ &= w_2(c) \cdot [\mathcal{O}_{X'}] \\ &= -\chi(c \cdot h_1 \cdot h_2 \cdot (a_1 h_1)) h_2 \cdot (a_1 h_1 - \binom{a_1}{2} h_1^2) \\ &\quad + \chi(c \cdot h_2 \cdot (a_1 h_1 - \binom{a_1}{2} h_1^2)) h_1 \cdot h_2 \cdot (a_1 h_1) \\ &\equiv -a_1 r d_{1,1,2} (a_1 h_1 \cdot h_2 - \binom{a_1}{2} h_1^2 \cdot h_2) \\ &\quad + a_1 d_{1,1,2} (a_1 \chi(c \cdot h_1 \cdot h_2) - \binom{a_1}{2} r d_{1,1,2}) [\mathcal{O}_x] \\ &\equiv a_1^2 d_{1,1,2} (-r h_1 \cdot h_2 + \chi(c \cdot h_1 \cdot h_2) [\mathcal{O}_x]) \\ &= a_1^2 d_{1,1,2} \widehat{u}_2(c). \end{aligned}$$

Let now \mathcal{F} be an S -flat family of torsion-free sheaves on X with fixed determinant bundle and such that the restriction $\mathcal{F}' := \mathcal{F}|_{S \times X'}$ remains flat over S and that the sequence

$$0 \rightarrow \mathcal{F}(-X_1) \rightarrow \mathcal{F} \rightarrow \mathcal{F}' \rightarrow 0$$

is exact. Then this exact sequence and the above numerical equivalences (3.6) and (3.7) induce isomorphisms of the following determinant line bundles on S :

$$\begin{aligned} \lambda_{\mathcal{F}'}(u_1(c|_{X'}))^{a_1 d_{1,1,2}} &\cong \lambda_{\mathcal{F}'}(w_2(c)|_{X'}) \\ &= \lambda_{\mathcal{F}}(w_2(c)) \otimes (\lambda_{\mathcal{F}(-X')}(w(c)))^{-1} \\ (3.8) \quad &\cong \lambda_{\mathcal{F}}(w_2(c) - w_2(c) \cdot [\mathcal{O}(-X')]) \\ &\cong (\lambda_{\mathcal{F}}(\widehat{u}_2(c)))^{a_1^2 d_{1,1,2}}. \end{aligned}$$

Note that as in [HL10, p. 219] fixing the determinant of the family \mathcal{F} leads to isomorphisms of determinant line bundles $\lambda_{\mathcal{F}}(D_0) \cong \lambda_{\mathcal{F}}(D_1)$ associated to numerically equivalent zero dimensional classes $D_0 \equiv D_1$.

Step 2: The second restriction: Since by assumption the further restriction $\mathcal{F}'' := \mathcal{F}'|_{S \times X''} = \mathcal{F}|_{S \times X''}$ to $S \times X''$ is also flat over S , and the corresponding sequence

$$0 \rightarrow \mathcal{F}'(-X'') \rightarrow \mathcal{F}' \rightarrow \mathcal{F}'' \rightarrow 0.$$

is exact, taking $H_2|_{X'}$ as polarisation on X' one obtains as in [HL10, eq. (8.2)] that

$$(3.9) \quad (\lambda_{\mathcal{F}''}(u_0(c|_{X''})))^{a_2 d_{1,2,2}} \cong (\lambda_{\mathcal{F}'}(u_1(c|_{X'})))^{a_2^2 d_{1,2,2}}.$$

End of proof: Combining the two formulas (3.8) and (3.9) we deduce

$$(\lambda_{\mathcal{F}''}(u_0(c|_{X''})))^{a_1 a_2 d_{1,1,2} d_{1,2,2}} \cong (\lambda_{\mathcal{F}}(\widehat{u}_2(c)))^{a_1^2 a_2^2 d_{1,1,2} d_{1,2,2}},$$

as asserted. \square

3.3. Semiampleness of determinant line bundles. In this section we prove the crucial semiampleness statement, which will later allow us to define the desired moduli space as the Proj-scheme associated with some finitely generated ring of invariant sections.

3.3.1. Setup. Let X be a smooth projective threefold, $H_1, H_2 \in \text{Pic}(X)$ two ample divisors, $c_1 \in H^2(X, \mathbb{Z})$, $c_2 \in H^4(X, \mathbb{Z})$, $c_3 \in H^6(X, \mathbb{Z})$ three classes, r a positive real number, $c \in K(X)_{\text{num}}$ a class with rank r , and Chern classes $c_j(c) = c_j$, and Λ a line bundle on X with $c_1(\Lambda) = c_1 \in H^2(X, \mathbb{Z})$.

Recall from Section 2.2 that the family of (H_1, H_2) -semistable sheaves of class c (and determinant Λ) is bounded, so that for sufficiently large $m \in \mathbb{N}$, each μ -semistable sheaf of class c is m -regular with respect to some chosen ample line bundle $\mathcal{O}_X(1)$, cf. [HL10, Lem. 1.7.2]. In particular, for each such sheaf F , the m -th twist $F(m)$ is globally generated with $h^0(F(m)) = P(m)$,

see for example [Laz04, Thm. 1.8.3]. Setting $V := \mathbb{C}^{P(m)}$ and $\mathcal{H} := V \otimes \mathcal{O}_X(-m)$, we obtain a surjection $\rho: \mathcal{H} \rightarrow F$ by composing the evaluation map $H^0(F(m)) \otimes \mathcal{O}_X(-m)$ with an isomorphism $V \rightarrow H^0(F(m))$. The sheaf morphism ρ defines a closed point

$$[q: \mathcal{H} \rightarrow F] \in \text{Quot}(\mathcal{H}, P(m))$$

in the Quot-scheme of quotients of \mathcal{H} with Hilbert polynomial $P(m)$.

Let $R^{\mu ss} \subset \text{Quot}(\mathcal{H}, P(m))$ be the locally closed subscheme of all quotients $[q: \mathcal{H} \rightarrow F]$ with class c and determinant Λ such that

- (i) F is μ -semistable of rank r , and
- (ii) ρ induces an isomorphism $V \xrightarrow{\cong} H^0(F(m))$.

The reductive group $SL(V)$ acts on $R^{\mu ss}$ by change of base in the vector space $H^0(F(m))$. This group action can be lifted to the reduction $R_{red}^{\mu ss}$, making the reduction morphism $R_{red}^{\mu ss} \rightarrow R^{\mu ss}$ equivariant with respect to the two $SL(V)$ -actions. Lifting the action one step further, we note that $SL(V)$ also acts on the weak normalisation

$$(3.10) \quad S := (R_{red}^{\mu ss})^{wn}$$

of $R_{red}^{\mu ss}$ in such a way that the weak normalisation morphism $(R_{red}^{\mu ss})^{wn} \rightarrow R_{red}^{\mu ss}$ intertwines the two $SL(V)$ -actions.

Let $\rho: \mathcal{O}_S \otimes \mathcal{H} \rightarrow \mathcal{F}$ denote the pullback of the universal quotient from $\text{Quot}(\mathcal{H}, P(m))$ to S . Choosing a fixed base point $x \in X$, as in Section 3.2.1 we consider the class $\hat{u}_2(c) := -rh_1 \cdot h_2 + \chi(c \cdot h_1 \cdot h_2)[\mathcal{O}_x] \in K(X)$, and the corresponding determinant line bundle

$$(3.11) \quad \mathcal{L}_2 := \lambda_{\mathcal{F}}(\hat{u}_2(c))$$

on the parameter space S .

Remark 3.4. Since by assumption all the sheaves parametrised by S have the same determinant Λ , it follows from the argument in [HL10, Ex. 8.1.8 ii)] that \mathcal{L}_2 is in fact independent of the chosen point $x \in X$, i.e., it is naturally induced by the classes of the two ample divisors H_1 and H_2 .

3.3.2. Semiampleness. The following is the main result of this section and the core ingredient in the construction of the moduli space.

Theorem 3.5 (Equivariant semiampleness). *There exists a positive integer $\nu \in \mathbb{N}$ such that $\mathcal{L}_2^{\otimes \nu}$ is generated over S by $SL(V)$ -invariant sections.*

Proof. Throughout the proof we consider a fixed a point $s \in S$. By Semistable Restriction (see Section 2.2), there exist positive natural numbers $a_1, a_2 \in \mathbb{N}$ such that

- (i) $a_1 H_1$ and $a_2 H_2$ are very ample and

- (ii) for any general (smooth) complete intersection curve X'' of elements in $|a_1H_1|$ and $|a_2H_2|$ the restriction $\mathcal{F}_s|_{X''}$ is semistable.

Remark 3.6. Concerning the second point note that on the curve X'' the notion of “semistability” is well-defined without fixing a further parameter.

Hence, it follows from Lemma 3.1 (applied to the very ample line bundles a_1H_1 and a_2H_2) and Lemma 2.7 that in order to prove the existence of sections in powers of \mathcal{L}_2 we may assume without loss of generality to be in the following setup:

Setup. *There exists a complete intersection curve X'' obtained by intersecting a general member X' of $|a_1H_1|$ with a general element of $|a_2H_2|$ such that the following holds: if we set $\mathcal{F}' := \mathcal{F}|_{S \times X'}$ and $\mathcal{F}'' := \mathcal{F}|_{S \times X''}$, then*

- (i) *both families \mathcal{F}' and \mathcal{F}'' are S -flat,*
- (ii) *we have two exact sequences of S -flat sheaves*

$$(3.12) \quad 0 \rightarrow \mathcal{F}(-X') \rightarrow \mathcal{F} \rightarrow \mathcal{F}' \rightarrow 0, \quad \text{and}$$

$$(3.13) \quad 0 \rightarrow \mathcal{F}'(-X'') \rightarrow \mathcal{F}' \rightarrow \mathcal{F}'' \rightarrow 0,$$

and

- (iii) *the restricted sheaf $\mathcal{F}_s|_{X''} \cong \mathcal{F}_s''$ is semistable.*

For this we have replaced S by $S \setminus T$, where T is the “non-flatness” locus from Lemma 3.1. Note that we may consider there $S_1 = \{s\}$, which implies that $s \notin T$.

For any class $c \in K(X)_{\text{num}}$ let $c'' = i_{X''}^*(c)$ denote the restriction to X'' , and let $P'' = P(c'')$ be the associated Hilbert polynomial with respect to the ample bundle $\mathcal{O}_{X''}(1) := \mathcal{O}_X(1)|_{X''}$.

Let m'' be a large positive integer, set $V'' := \mathbb{C}^{P''(m'')}$, $\mathcal{H}'' := V'' \otimes \mathcal{O}_{X''}(-m'')$, and let $Q_{X''}$ be the closed subscheme of $\text{Quot}(\mathcal{H}'', P'')$ parametrising quotients with determinant $\Lambda|_{X''}$. Denote by $\mathcal{O}_{Q_{X''}} \otimes \mathcal{H}'' \rightarrow \widehat{\mathcal{F}}''$ the universal quotient sheaf and set

$$(3.14) \quad \mathcal{L}_0'' := \lambda_{\widehat{\mathcal{F}}''}(u_0(c'')).$$

Note that on the curve X'' slope-semistability and Gieseker-semistability coincide, cf. Remark 3.6. As in [HL10, p. 223], the following lemma hence follows from the construction of the moduli space for semistable sheaves on the curve X'' .

Lemma 3.7. *After increasing m'' if necessary, the following holds:*

- (1) *For a given point $[q : \mathcal{H}'' \rightarrow E] \in Q_{X''}$ the following assertions are equivalent:*
 - (i) *The quotient E is a semistable sheaf and the induced map $V'' \rightarrow H^0(X, E(m''))$ is an isomorphism.*

- (ii) The point $[q] \in Q_{X''}$ is GIT-semistable with respect to the natural $SL(V'')$ -linearisation of \mathcal{L}_0'' .
- (iii) There exists a positive integer $\nu \in \mathbb{N}$ and an $SL(V'')$ -invariant section $\sigma \in H^0(Q_{X''}, \mathcal{L}_0'')$ such that $\sigma([q]) \neq 0$.
- (2) Two points $[q_i : \mathcal{H}'' \rightarrow E_i] \in Q_{X''}$, $i = 1, 2$, are separated by invariant sections in some tensor power of \mathcal{L}_0'' if and only if either one of them is semistable but the other is not, or both points are semistable but E_1 and E_2 are not S -equivalent.

In addition to assertions (1) and (2) of the previous lemma, by increasing m'' further if necessary, we may assume that for each $s \in S$ the restricted sheaf \mathcal{F}_s'' is m'' -regular with respect to $\mathcal{O}_{X''}(1)$. Consequently, each such sheaf is globally generated and defines a closed point in $Q_{X''}$ with the additional property that the induced map from $V'' \rightarrow H^0(X'', \mathcal{F}_s''(m''))$ is an isomorphism.

Denoting the projection from $S \times X''$ to the first factor by p , the push-forward $p_*(\mathcal{F}''(m''))$ is a locally free $SL(V)$ -linearised \mathcal{O}_S -sheaf of rank $P''(m'')$ on S . The associated $SL(V)$ -equivariant projective frame bundle $\pi: \tilde{S} \rightarrow S$ parametrises a quotient

$$\mathcal{O}_{\tilde{S}} \otimes \mathcal{H}'' \rightarrow \pi^* \mathcal{F}'' \otimes \mathcal{O}_{\pi}(1),$$

which induces an $SL(V'')$ -invariant morphism $\Phi'': \tilde{S} \rightarrow Q_{X''}$ that is compatible with the $SL(V)$ -action on \tilde{S} . We summarise our situation in the following diagram:

$$(3.15) \quad \begin{array}{ccc} \tilde{S} & \xrightarrow{\Phi''} & Q_{X''} \\ \pi \downarrow & & \\ S & & . \end{array}$$

Setting $k_0 := a_1 a_2 d_{1,1,2}$ and $k_2 := a_1^2 a_2^2 d_{1,1,2} d_{1,2,2}$, just rewriting equality (3.5) of Proposition 3.3 we obtain

$$(3.16) \quad \lambda_{\mathcal{F}''}(u_0(c''))^{\otimes k_0} \cong \lambda_{\mathcal{F}}(\hat{u}_2)^{\otimes k_2}.$$

With these preparations in place, we compute:

$$\begin{aligned} (\Phi'')^*(\mathcal{L}_0'')^{\otimes k_0} &= (\Phi'')^*(\lambda_{\hat{\mathcal{F}}''}(u_0(c'')))^{\otimes k_0} && \text{by definition, see eq. (3.14)} \\ &\cong \lambda_{\pi^* \mathcal{F}'' \otimes \mathcal{O}_{\pi}(1)}(u_0(c''))^{\otimes k_0} && \text{by [HL10, Lem. 8.1.2 ii)]} \\ &\cong \lambda_{\pi^* \mathcal{F}''}(u_0(c''))^{\otimes k_0} && \text{by [HL10, Lem. 8.1.2 ii)]} \\ &\cong \pi^* \lambda_{\mathcal{F}''}(u_0(c''))^{\otimes k_0} && \text{by [HL10, Lem. 8.1.2 iv)]} \\ &\cong \pi^* \lambda_{\mathcal{F}}(\hat{u}_2)^{\otimes k_2} && \text{by eq. (3.16)} \\ &= \pi^*(\mathcal{L}_2)^{\otimes k_2} && \text{by definition, see eq. (3.11).} \end{aligned}$$

Let σ be a $SL(V'')$ -invariant section in $(\mathcal{L}_0'')^{\otimes \nu k_0}$ that does not vanish at a given point of the form $[q: \mathcal{H}_t'' \rightarrow \mathcal{F}_t|_{X''}]$. Since Φ'' is $SL(V)$ -invariant, the pullback $(\Phi'')^*(\sigma)$ is an $SL(V'') \times SL(V)$ -invariant section in $(\Phi'')^*(\mathcal{L}_0'')^{\otimes k_0} \cong \pi^* \lambda_{\mathcal{F}}(\hat{u}_2)^{\otimes \nu k_2}$. Since π is a good quotient of \tilde{S} by the $SL(V'')$ -action, $(\Phi'')^*(\sigma)$ descends to an $SL(V)$ -invariant section $l_{\mathcal{F}}(\sigma) \in H^0(S, (\mathcal{L}_2)^{\otimes \nu k_2})$ that does not vanish at $t \in S$.

Finally, recall that we want to produce a section in \mathcal{L}_2 that does not vanish at our given point $s \in S$. As $\mathcal{F}_s|_{X''}$ is semistable and the induced map $V'' \rightarrow H^0(X'', \mathcal{F}_s''(m''))$ is an isomorphism, by Lemma 3.7(1) there exists a positive integer $\nu \in \mathbb{N}$ and a section $\sigma \in H^0(Q_{X''}, (\mathcal{L}_0'')^{\nu k_0})^{SL(V'')}$ such that the induced section $l_{\mathcal{F}}(\sigma) \in H^0(S, (\mathcal{L}_2)^{\nu k_2})^{SL(V)}$ fulfills $l_{\mathcal{F}}(\sigma)(s) \neq 0$. This completes the proof of Theorem 3.5. \square

Remark 3.8. In the situation of the setup of the proof of Theorem 3.5 we have thus shown the existence of a linear map

$$l_{\mathcal{F}} : H^0(Q_{X''}, (\mathcal{L}_0'')^{\nu k_0})^{SL(V'')} \rightarrow H^0(S, (\mathcal{L}_2)^{\nu k_2})^{SL(V)}$$

with the property that for any pair of points $s_1, s_2 \in S$ and $q_1, q_2 \in Q_C$ which correspond to one another through diagram (3.15) and any $\sigma \in H^0(Q_{X''}, (\mathcal{L}_0'')^{\nu k_0})^{SL(V'')}$, one has

- (1) $l_{\mathcal{F}}(\sigma)(s_1) = 0$ if and only if $\sigma(q_1) = 0$, and
- (2) $l_{\mathcal{F}}(\sigma)(s_1) \neq l_{\mathcal{F}}(\sigma)(s_2)$ if $\sigma(q_1) \neq \sigma(q_2)$.

4. A PROJECTIVE MODULI SPACE FOR SLOPE-SEMISTABLE SHEAVES

In this section we will give the construction of the “moduli space” of μ -semistable sheaves. We continue to use the notation introduced in the previous section, see especially Section 3.3.1. We have seen in Theorem 3.5 that for $\nu \in \mathbb{N}$ big enough, the line bundle $\mathcal{L}_2^{\otimes \nu}$ is generated by $SL(V)$ -invariant sections. Hence, it is a natural idea to construct the moduli space as an image of S under the map given by invariant sections of $\mathcal{L}_2^{\otimes \nu}$ for $\nu \gg 0$. For this, we set

$$W_{\nu} := H^0(S, \mathcal{L}_2^{\otimes \nu})^{SL(V)}.$$

Since S is Noetherian, for every $\nu \in \mathbb{N}$ such that W_{ν} generates $\mathcal{L}_2^{\otimes \nu}$ over S , there exists a finite-dimensional \mathbb{C} -vector subspace \hat{W}_{ν} of W_{ν} that still generates $\mathcal{L}_2^{\otimes \nu}$ over S . We consider the induced $SL(V)$ -invariant morphism $\varphi_{\hat{W}_{\nu}} : S \rightarrow \mathbb{P}(\hat{W}_{\nu}^*)$ and set

$$M_{\hat{W}_{\nu}} := \varphi_{\hat{W}_{\nu}}(S).$$

We will see in the next section that $M_{\hat{W}_{\nu}}$ is a projective variety.

4.1. Compactness. The key to proving the compactness of our yet to be constructed moduli spaces lies in the following generalisation of Langton's Theorem to the case of multipolarisations:

Theorem 4.1 (Langton's Theorem). *Let $\mathcal{R} \supset k$ be a discrete valuation ring with field of fractions K , let $i: X \times \operatorname{Spec} K \rightarrow X \times \operatorname{Spec} \mathcal{R}$ be the inclusion of the generic fibre, and let $j: X_k \rightarrow X \times \operatorname{Spec} \mathcal{R}$ be the inclusion of the closed fibre in $X \times \operatorname{Spec} \mathcal{R}$ over $\operatorname{Spec} \mathcal{R}$. Then, for any (H_1, H_2) -semistable torsion-free coherent sheaf E_K over $X \times \operatorname{Spec} K$, there exists a torsion-free coherent sheaf over $X \times \operatorname{Spec} \mathcal{R}$ such that $i^*E \cong E_K$ and such that j^*E is torsion-free and semistable.*

Proof. The proof of Langton's completeness result [Lan75] (for slope-functions defined by a single integral ample divisor) literally works for more general slope-functions as introduced in Definition 2.3. The key point is to note that degrees with respect to multipolarisations also can be seen as coefficients of appropriate terms in some Hilbert polynomials; cf. [Kle66, p.296]. \square

By replacing Langton's original theorem with Theorem 4.1, the following result can now be obtained using exactly the same arguments as in [HL10, Prop. 8.2.5].

Proposition 4.2. *Let $Z \subset S$ be any $SL(V)$ -invariant closed subvariety. If T is a separated scheme of finite type over \mathbb{C} , and if $\varphi: Z \rightarrow T$ is any $SL(V)$ -invariant morphism, then the image $\varphi(Z) \subset T$ is complete. In particular, any $SL(V)$ -invariant function on S is constant.*

4.2. Construction. With the results of the previous section in place, it is now a natural idea to consider the projective varieties $M_{\hat{W}_\nu}$ for increasing values of ν . Following this approach, and applying exactly the same reasoning as in the proof of [HL10, Prop. 8.2.6] we obtain the following result.

Proposition 4.3 (Finite generation). *There exists an integer $N > 0$ such that the graded ring $\bigoplus_{k \geq 0} W_{kN}$ is generated over $W_0 = \mathbb{C}$ by finitely many elements of degree one.*

Using the previous finite generation result, we can now define our desired moduli space.

Definition 4.4 (Moduli space for slope-semistable sheaves). *Let $N \geq 1$ be a natural number with the properties spelled out in Proposition 4.3 above. Then, we define the polarised variety $(M^{\mu ss}, L)$ to be the projective variety*

$$M^{\mu ss} := M^{\mu ss}(c, \Lambda) := \operatorname{Proj} \bigoplus_{k \geq 0} H^0(S, \mathcal{L}_2^{\otimes kN})^{SL(V)},$$

together with the ample line bundle $L := \mathcal{O}_{M^{\mu ss}}(1)$. Moreover, we let $\Phi: S \rightarrow M^{\mu ss}$ be the induced $SL(V)$ -invariant morphism with $\Phi^*(L) = \mathcal{L}_2^{\otimes N}$.

4.3. Functorial behaviour. While the projective variety $M^{\mu ss}$ is not a coarse moduli space in general, it nevertheless has certain universal properties, which we describe next. We start by formulating and proving a result that is slightly weaker than the Main Theorem. Subsequently, we will show that by choosing the “correct” polarising line bundle for $M^{\mu ss}$, we obtain the universal properties stated in the Main Theorem.

Proposition 4.5. *Let $\underline{M}^{\mu ss}$ denote the functor that associates to a weakly normal variety S the set of isomorphism classes of S -flat families of μ -semistable sheaves of class c and determinant Λ . Then, there is a natural transformation from $\underline{M}^{\mu ss}$ to $\underline{Hom}(\cdot, M^{\mu ss})$, mapping a family \mathcal{F} to a classifying morphism $\Phi_{\mathcal{F}}$, with the following properties:*

- (1) *For any B -flat family \mathcal{F} of μ -semistable sheaves of class c and determinant Λ with induced classifying morphism $\Phi_{\mathcal{F}}: B \rightarrow M^{\mu ss}$ we have*

$$(4.1) \quad \Phi_{\mathcal{F}}^*(L) \cong \lambda_{\mathcal{F}}(\hat{u}_2)^N,$$

where $\lambda_{\mathcal{F}}(\hat{u}_2)$ is the determinant line bundle on S induced by \mathcal{F} and \hat{u}_2 ; cf. Section 2.1.

- (2) *For any other triple of a natural number N' , a projective variety M' , and an ample line bundle L' fulfilling the conditions spelled out in (1), there exist a natural number $d \in \mathbb{N}^{>0}$, and a uniquely determined morphism $\psi: M^{\mu ss} \rightarrow M'$ such that $\psi^*(L')^{\otimes dN} \cong L^{\otimes dN'}$.*

In the subsequent proofs we will use the following standard terminology.

Definition 4.6. *Let X be a proper variety, and L an line bundle on X . Then, the section ring of L is defined to be*

$$R(X, L) := \bigoplus_{k \geq 0} H^0(X, L^{\otimes k}).$$

For any $d \geq 2$, we define the d -th Veronese subring of $R(X, L)$ to be

$$R(X, L)_{(d)} := \bigoplus_{d|k} H^0(X, L^{\otimes k}) \subset R(X, L).$$

Proof of Proposition 4.5. In the proof of part (1) we follow [HL10, proof of Lem. 4.3.1]. We will use the notation introduced in Section 3.3.1. Let B be a weakly normal variety and \mathcal{F} a B -flat family of μ -semistable sheaves with Hilbert polynomial P with respect to the chosen ample polarisation $\mathcal{O}_X(1)$. Denote the natural projections of $B \times X$ by $p: B \times X \rightarrow B$ and $q: B \times X \rightarrow X$. Let $m \in \mathbb{N}$ as in Section 3.3.1, such that each semistable sheaf F with the given invariants is m -regular. We set $V := \mathbb{C}^{\oplus P(m)}$ and $\mathcal{H} := V \otimes \mathcal{O}_X(-m)$. The

sheaf $V_{\mathcal{F}} := p_*(\mathcal{F} \otimes q^*\mathcal{O}_X(m))$ is locally free of rank $P(m)$, and there is a canonical surjection

$$\varphi_{\mathcal{F}}: p^*V_{\mathcal{F}} \otimes q^*\mathcal{O}_X(-m) \twoheadrightarrow \mathcal{F}.$$

Let $\pi: R(\mathcal{F}) \rightarrow B$ be the frame bundle associated with $V_{\mathcal{F}}$; the group $GL(V)$ acts on $R(\mathcal{F})$ making it a $GL(V)$ -principal bundle with good quotient π . Since the pullback of $V_{\mathcal{F}}$ to $R(\mathcal{F})$ has a universal trivialisation, we obtain a canonically defined quotient

$$\tilde{q}_{\mathcal{F}}: \mathcal{O}_{R(\mathcal{F})} \otimes_{\mathbb{C}} \mathcal{H} \twoheadrightarrow (\pi \times \text{id}_X)^*\mathcal{F}$$

on $R(\mathcal{F}) \times X$, giving rise to a classifying morphism

$$\tilde{\Psi}_{\mathcal{F}}: R(\mathcal{F}) \rightarrow \text{Quot}(\mathcal{H}, P).$$

The map $\tilde{\Psi}_{\mathcal{F}}$ is equivariant with respect to the $GL(V)$ -actions on $R(\mathcal{F})$ and $R^{\mu ss}$, where the latter action is induced by the $SL(V)$ -action via $PGL(V)$, cf. [HL10, Lem. 4.3.2]. Since the sheaf \mathcal{F}_b was assumed to be slope-semistable for all $b \in B$, the image of $\tilde{\Psi}_{\mathcal{F}}$ is contained in $R^{\mu ss}$. Note that as a principal bundle over the seminormal variety B , the space $R(\mathcal{F})$ is itself seminormal. Consequently, by Proposition 2.6 the map $\tilde{\Psi}_{\mathcal{F}}$ lifts to a morphism from $R(\mathcal{F})$ to $(R^{\mu ss})^{wn} = S$, which we will continue to denote by $\tilde{\Psi}_{\mathcal{F}}$.

Composing $\tilde{\Psi}_{\mathcal{F}}$ with the $SL(V)$ -invariant morphism $\Phi: S \rightarrow M^{\mu ss}$, we obtain a $GL(V)$ -invariant morphism $\tilde{\Phi}_{\mathcal{F}}: R(\mathcal{F}) \rightarrow M^{\mu ss}$. Since π is a good quotient, and hence in particular a categorical quotient, there exists a uniquely determined morphism $\Phi_{\mathcal{F}}: B \rightarrow M^{\mu ss}$ such that the following diagram commutes:

$$(4.2) \quad \begin{array}{ccc} R(\mathcal{F}) & \xrightarrow{\tilde{\Psi}_{\mathcal{F}}} & S \\ \pi \downarrow & \searrow \tilde{\Phi}_{\mathcal{F}} & \downarrow \Phi \\ B & \xrightarrow{\Phi_{\mathcal{F}}} & M^{\mu ss}. \end{array}$$

Assigning $\Phi_{\mathcal{F}} \in \text{Mor}(B, M^{\mu ss})$ to \mathcal{F} yields the desired natural transformation $\underline{M}^{\mu ss} \rightarrow \underline{\text{Hom}}(\cdot, M^{\mu ss})$. It remains to show the isomorphism of line bundles (4.1). For this we note that

$$\pi^*\Phi_{\mathcal{F}}^*\mathcal{O}_{M^{\mu ss}}(1) \cong \tilde{\Psi}_{\mathcal{F}}^*(\mathcal{L}_2^{\otimes N}) \cong \lambda_{\pi^*\mathcal{F}}(\hat{u}_2)^{\otimes N} \cong \pi^*\lambda(\hat{u}_2)^{\otimes N}.$$

Since $\pi^*: \text{Pic}(B) \rightarrow \text{Pic}(R(\mathcal{F}))$ is injective by [MFK94, Ch. 1, §3, Prop. 1.4], cf. [LP92, Lem. 2.14], we obtain

$$\Phi_{\mathcal{F}}^*\mathcal{O}_{M^{\mu ss}}(1) \cong \lambda(\hat{u}_2)^{\otimes N},$$

as claimed in (1).

Next, we prove (2). Let (M', L', N') be another triple satisfying the conditions of (1), then applying the universal property we obtain a uniquely determined morphism $\Phi': S \rightarrow M'$ such that $(\Phi')^*(L') \cong \mathcal{L}_2^{\otimes N'}$. We claim that

Φ' is $SL(V)$ -invariant. Indeed, the restriction of Φ' to an $SL(V)$ -orbit \mathcal{O} gives the classifying map for the restriction of the universal family to \mathcal{O} . However, this latter classifying map is constant, which implies that $\Phi'(\mathcal{O}) = \{\text{pt.}\}$, as claimed.

Let $d \in \mathbb{N}^{>0}$ such that the d -th Veronese subring $R(M', L')_{(d)}$ of $R(M', L')$ is generated in degree one. We denote by $f: M' \rightarrow \text{Proj}(R(M', L')_{(d)})$ the natural isomorphism. With this notation, the pullback $f^*(\mathcal{O}(1))$ is naturally isomorphic to $(L')^{\otimes d}$. Setting $d' := dN'$ and using the universal property of L' , we obtain a natural morphism of graded rings

$$R(M', L')_{(dN)} \rightarrow \bigoplus_{d'|k} H^0(S, \mathcal{L}_2^{\otimes kN})^{SL(V)}$$

by pulling back sections via the $SL(V)$ -invariant map Φ' . This in turn induces a uniquely determined morphism

$$\psi: M = \text{Proj}\left(\bigoplus_{d'|k} H^0(S, \mathcal{L}_2^{\otimes kN})^{SL(V)}\right) \rightarrow \text{Proj}(R(M', L')_{(dN)}) = M'$$

such that $\psi^*(L')^{\otimes dN} = L^{\otimes dN'}$. This concludes the proof of (2). \square

Corollary 4.7 (Weak normality). *The variety $M^{\mu ss}$ is weakly normal.*

Proof. It suffices to notice that the weak normalisation $\nu: (M^{\mu ss})^{wn} \rightarrow M^{\mu ss}$ together with the line bundle $L' := \nu^*L$ and the natural number N has the universal properties spelled out in part (1) of Proposition 4.5, since by Proposition 2.6 every map from a weakly normal variety S to $M^{\mu ss}$ can be lifted in a unique way to a map from S to $(M^{\mu ss})^{wn}$ satisfying the required pullback properties. Consequently, part (2) of Proposition 4.5 yields a uniquely determined morphism $\psi: M^{\mu ss} \rightarrow (M^{\mu ss})^{wn}$, which gives an inverse to ν . Hence, $M^{\mu ss}$ is weakly normal, as claimed. \square

Comparing the universal property (2) of Proposition 4.5 with the one claimed in the Main Theorem, we see that we need to improve the uniqueness statement in Proposition 4.5(2). This amounts to showing that although the map $\Phi: S \rightarrow M^{\mu ss}$ is not proper, owing to its $SL(V)$ -invariance it still has certain properties that are reminiscent of the Stein fibration and Iitaka fibration for semiample line bundles on proper varieties, cf. [Laz04, Sect. 2.1.C].

We first show that while the map $\Phi: S \rightarrow M^{\mu ss}$ might not be a categorical quotient, it nevertheless displays the “correct” behaviour with respect to invariant regular functions.

Lemma 4.8. *Let $\Phi: S \rightarrow M^{\mu ss}$ be as before. Then, we have*

$$\Phi_*(\mathcal{O}_S)^{SL(V)} = \mathcal{O}_{M^{\mu ss}}.$$

Proof. We first notice the following generalisation of the classical projection formula [Har77, Chap. 2, Ex. 5.1(d)] to the equivariant setting, cf. [Gre10, Lem. 9.2]:

Lemma 4.9 (Equivariant projection formula). *Let G be an algebraic group, let Y be an algebraic G -variety, and let $f: Y \rightarrow Z$ be a G -invariant morphism to an algebraic variety Z . Then, for any G -linearised coherent algebraic sheaf \mathcal{F} on Y , and any locally free sheaf \mathcal{G} of finite rank on Z , there is a natural isomorphism*

$$f_*(\mathcal{F} \otimes f^*\mathcal{G})^G \cong f_*(\mathcal{F})^G \otimes \mathcal{G}.$$

Here, $f^*\mathcal{G}$ is given the natural G -linearisation as a pull-back bundle via an invariant morphism, and $\mathcal{F} \otimes f^*\mathcal{G}$ is given the natural tensor product linearisation. \square

We proceed as follows. Since Φ is invariant, we obtain a natural morphism $\eta: \mathcal{O}_{M^{\mu ss}} \rightarrow \Phi_*(\mathcal{O}_S)^{SL(V)}$ of quasi-coherent sheaves of $\mathcal{O}_{M^{\mu ss}}$ -modules. Since $L = \mathcal{O}_{M^{\mu ss}}(1)$ is ample and $R(M^{\mu ss}, L)$ is generated in degree one, in order for η to be an isomorphism, it suffices to show that the induced map

$$\hat{\eta}_k: H^0(M^{\mu ss}, L^{\otimes k}) \rightarrow H^0(M^{\mu ss}, \Phi_*(\mathcal{O}_S)^{SL(V)} \otimes L^{\otimes k})$$

is an isomorphism for all natural numbers $k \geq 1$, cf. [Har77, Prop. II.5.15]. For this, we note that $\hat{\eta}_k$ can be factored in the following way:

$$\begin{aligned} H^0(M^{\mu ss}, L^{\otimes k}) &\xrightarrow{\alpha_k} H^0(S, \mathcal{L}_2^{\otimes kN})^{SL(V)} \xrightarrow{\beta_k} H^0(M^{\mu ss}, \Phi_*(\mathcal{O}_S \otimes \Phi^*L^{\otimes k})^{SL(V)}) \\ &\xrightarrow{\gamma_k} H^0(M^{\mu ss}, \Phi_*(\mathcal{O})^{SL(V)} \otimes L^{\otimes k}). \end{aligned}$$

In the previous diagram, α_k and β_k are isomorphisms by definition and by the $SL(V)$ -equivariant isomorphism $\Phi^*L^{\otimes k} \cong \mathcal{L}_2^{\otimes kN}$, and γ_k is an isomorphism by the equivariant projection formula, Lemma 4.9 above. Consequently, the composition, which is equal to $\hat{\eta}_k$, is an isomorphism. This concludes the proof of Lemma 4.8.

The following is the analogue of [Laz04, Ex. 2.1.14] in our equivariant setup.

Lemma 4.10 (Injectivity of equivariant pull-back). *The natural pull-back map*

$$\Phi^*: \text{Pic}(M^{\mu ss}) \rightarrow \text{Pic}_{SL(V)}(S)$$

from $\text{Pic}(M^{\mu ss})$ to the group of $SL(V)$ -linearised line bundles on S is injective.

Proof. Let D be a line bundle on $M^{\mu ss}$ such that $\Phi^*(B)$ is $SL(V)$ -equivariantly isomorphic to the trivial line bundle with the trivial $SL(V)$ -linearisation. We

note that any constant function on S is $SL(V)$ -invariant and apply the equivariant projection formula, Lemma 4.9, as well as Lemma 4.8 to obtain the following chain of isomorphisms

$$\begin{aligned} H^0(S, \mathcal{O}_S)^{SL(V)} &\cong H^0(M^{\mu ss}, \Phi_*(\Phi^*D)^{SL(V)}) \\ &\cong H^0(M^{\mu ss}, \Phi_*(\mathcal{O}_S)^{SL(V)} \otimes D) \\ &\cong H^0(M^{\mu ss}, D), \end{aligned}$$

and hence a section in $H^0(M^{\mu ss}, D)$ that does not vanish on any component of S . As the same reasoning applies to the dual D^{-1} , the line bundle D is trivial, as claimed. \square

Lemma 4.11 (Injectivity of pull-back). *The natural pull-back map*

$$\Phi^*: \text{Pic}(M^{\mu ss}) \rightarrow \text{Pic}(S)$$

is injective.

Proof. Since $SL(V)$ is semisimple, and therefore has no nontrivial characters, by [MFK94, Chap. 1, §3, Prop. 1.4] the forgetful map $\text{Pic}_{SL(V)}(S) \rightarrow \text{Pic}(S)$ is injective. Together with Lemma 4.10 this yields the claim. \square

We make a first improvement concerning the universal properties of Proposition 4.5.

Lemma 4.12. *Using the notation of Proposition 4.5, we may take $d = 1$. I.e., for any other triple of a natural number N' , a projective variety M' , and an ample line bundle L' fulfilling the conditions spelled out in part (1) of Proposition 4.5, there exist a uniquely determined morphism $\psi: M^{\mu ss} \rightarrow M'$ such that $\psi^*(L')^N \cong L^{\otimes N'}$. Call this universal property Property (2').*

Proof. Let (M', L', N') be another triple having property (1) of Proposition 4.5. Then, as in the proof of Proposition 4.5 let $\Phi': S \rightarrow M'$ be the classifying morphism for the universal family over S . Via the Proj construction, we obtain a uniquely determined morphism $\psi: M^{\mu ss} \rightarrow M'$ such that the following diagram commutes

$$\begin{array}{ccc} & S & \\ \Phi \swarrow & & \searrow \Phi' \\ M^{\mu ss} & \xrightarrow{\psi} & M'. \end{array}$$

By the universal properties of L and L' , we have natural isomorphisms

$$\Phi^*(\psi^*(L')^{\otimes N}) \cong (\Phi')^*(L')^{\otimes N} \cong \mathcal{L}_2^{\otimes NN'} \cong \Phi^*(L^{\otimes N'}).$$

As Φ^* is injective by Lemma 4.11, this implies $\psi^*(L')^{\otimes N} \cong L^{\otimes N'}$, which concludes the proof. \square

We are now in the position to choose an “optimal” line bundle on $M^{\mu ss}$ that has the universal property stated in the Main Theorem.

Proposition 4.13. *Let N be minimal such that $(M^{\mu ss}, L, N)$ has property (1) of Proposition 4.5 and Property (2’), cf. Lemma 4.12. Then, $(M \cong M^{\mu ss}, L, N)$ has the following universal property (2’): for any other triple of a natural number N' , a projective variety M' , and an ample line bundle L' fulfilling the conditions spelled out in part (1) of Proposition 4.5, one has $N|N'$ and there exist a uniquely determined morphism $\psi: M^{\mu ss} \rightarrow M'$ such that $\psi^*(L') \cong L^{\otimes \frac{N'}{N}}$.*

Proof. Consider another triple of a natural number N' , a projective variety M' , and an ample line bundle L' fulfilling the conditions spelled out in (1) of Proposition 4.5, and let $\psi: M \rightarrow M'$ be the uniquely determined morphism of Lemma 4.12. In order to establish the claim, it suffices to show that $N|N'$, as the rest follows by the same argument as in the proof of Lemma 4.12.

Suppose that N does not divide N' and let

$$(4.3) \quad e = \text{lcd}(N, N') < N$$

be their largest common divisor. There exist $a, b \in \mathbb{Z}$ such that $e = aN' + bN$. We set $A := \psi^*(L')^{\otimes a} \otimes L^{\otimes b}$. The pull-back of A to S via Φ equals $\mathcal{L}_2^{\otimes(aN'+bN)} = \mathcal{L}_2^{\otimes e}$ and using the injectivity of Φ^* provided by Lemma 4.11, we infer that $A^{\otimes \frac{N}{e}} \cong L$. Hence, A is ample. Aiming for a contradiction to the minimality of N , we claim that the triple (M, A, e) has the universal properties (1) and (2’).

Let B be a weakly normal variety and \mathcal{F} a B -flat family of μ -semistable sheaves of class c and determinant Λ on X . We have two classifying morphisms $\Phi_{\mathcal{F}}: B \rightarrow M = M^{\mu ss}$ and $\Phi'_{\mathcal{F}}: B \rightarrow M'$. The morphism $\psi: M \rightarrow M'$ given by property (2) for M is such that $\psi \circ \Phi_{\mathcal{F}} = \Phi'_{\mathcal{F}}$, as can be seen from the proof of Proposition 4.5. Hence,

$$\Phi_{\mathcal{F}}^*(A) = (\Phi'_{\mathcal{F}})^*((L')^{\otimes a}) \otimes \Phi_{\mathcal{F}}^*(L^{\otimes b}) \cong \lambda_{\mathcal{F}}(\hat{u}_2)^{aN'+bN} = \lambda_{\mathcal{F}}(\hat{u}_2)^e.$$

In order to prove (2’), let (M'', L'', N'') be any other triple having property (1). Then, as in the proof of Lemma 4.12 we have a natural commutative diagram

$$\begin{array}{ccc} & S & \\ \Phi \swarrow & & \searrow \Phi'' \\ M^{\mu ss} & \xrightarrow{\psi} & M'' \end{array}$$

By construction, we obtain natural isomorphisms

$$\Phi^*(A^{\otimes N''}) \cong \mathcal{L}_2^{\otimes eN''} \cong (\Phi'')^*(L'')^{\otimes e} \cong \Phi^*(\psi^*(L'')^{\otimes e})$$

on S . Using again injectivity of Φ^* , Lemma 4.11, we conclude that $A^{\otimes N''} \cong \psi^*(L''^{\otimes e})$.

In summary, we have established property (1) and (2') for the new triple (M, A, e) . This, together with the inequality (4.3) yields a contradiction to the minimality of N . Hence, we conclude $N|N'$, which was to be shown. \square

As an immediate consequence, we obtain in the usual way:

Corollary 4.14. *The triple $(M^{\mu ss}, L, N)$ with N minimal as above is uniquely determined up to isomorphism by the properties (1) and (2'').*

Remark 4.15. With Proposition 4.13 at hand, we have now established all claims made in the Main Theorem.

5. GEOMETRY OF THE MODULI SPACE

In this section we start to investigate the geometry of $M^{\mu ss}$. First, we look at the separation properties of the map $\Phi: S \rightarrow M^{\mu ss}$, and second, we prove that it provides a compactification for the moduli space of μ -stable reflexive sheaves.

5.1. Separation properties. The geometry of the map Φ will be studied in terms of Jordan-Hölder filtrations. Let us introduce the relevant terminology.

Proposition and Notation 5.1. *Let X be a projective threefold endowed with a multi-polarisation (H_1, H_2) with respect to which we consider slope-semistability. For a μ -semistable sheaf on X , there exists a μ -Jordan-Hölder filtration (in the sense of [HL10, Def. 1.5.1]). Let $gr^\mu F$ denote the graded sheaf associated with a μ -Jordan-Hölder filtration of F with torsion-free factors and set $F^\sharp := (gr^\mu F)^{\vee\vee}$. Then, F^\sharp is a reflexive μ -polystable sheaf on X , which depends only on F and not on the chosen Jordan-Hölder filtration.*

Proof. Existence of μ -Jordan-Hölder filtrations is shown exactly as in [HL10, Prop. 1.5.2], uniqueness as in [HL10, Cor. 1.6.10]. See also [Miy87]. \square

Notation 5.2. *For any μ -semistable sheaf F as above we consider the natural map $\iota: gr^\mu F \rightarrow F^\sharp$. Since $gr^\mu F$ is torsion-free, ι is injective, and the quotient sheaf $F^\sharp/gr^\mu F$ is supported in codimension at least two. We associate to F the one-dimensional support Chow-cycle of the sheaf $F^\sharp/gr^\mu F$, which we will denote by C_F .*

Remark 5.3. The cycle C_F depends only on F and not on the chosen Jordan-Hölder filtration, as can be seen by reduction to the surface case [HL10, Cor. 1.5.10] using repeated hyperplane sections.

The connection between Jordan-Hölder filtrations and restriction of semistable sheaves to curves is established by the following basic result.

Lemma 5.4. *Let F_1, F_2 be μ -semistable torsion-free sheaves on X . Then F_1^\sharp and F_2^\sharp are isomorphic if and only if their restrictions to a general intersection curve $X_1 \cap X_2$, $X_j \in |m_j H_j|$, for $m_j \gg 0$, are S -equivalent.*

Proof. Choose Jordan-Hölder filtrations with torsion-free factors for F_1 and F_2 . Next, choose a general complete intersection curve $X'' = X_1 \cap X_2$ with the following properties :

- (i) the curve X'' avoids the singularities of $gr^\mu F_1$ and $gr^\mu F_2$, and
- (ii) the restriction of the Jordan-Hölder filtrations of F_1 and F_2 to X'' are Jordan-Hölder filtrations for $F_1|_{X''}$ and $F_2|_{X''}$.

Item (i) is achievable, since torsion-free sheaves are locally free in codimension one [?, Cor. to Lem. 1.1.8]; i.e., their singularities lie in codimension two or higher. Item (ii) is achievable by Semistable Restriction, cf. Section 2.2.

Then, by item (ii) the restricted sheaves $F_1|_{X''}$ and $F_2|_{X''}$ are S -equivalent if and only if $(gr^\mu F_1)|_{X''} \cong (gr^\mu F_2)|_{X''}$, and this in turn is equivalent to $F_1^\sharp|_{X''} \cong F_2^\sharp|_{X''}$ by item (i).

Consider now the reflexive sheaf $E := \mathcal{H}om(F_1^\sharp, F_2^\sharp)$ on X . Since reflexive sheaves are locally free in codimension two [?, Lem. 1.1.10], and since moreover Serre duality holds in certain degrees for reflexive sheaves on projective threefolds, see [Har80, Rem. 2.5.1] or [BS76, Ch. IV, Thm. 3.1], we may choose $m_1 \gg 0$ and $X_1 \in |m_1 H_1|$, as well as $m_2 \gg 0$ and $X_2 \in |m_2 H_2|$ such that

- (α) X_1 and $X_1 \cap X_2$ are smooth,
- (β) X_1 avoids the singularities of E , and
- (γ) $H^1(X, E(-X_1)) = H^1(X_1, E_{X_1}(-X_2)) = \{0\}$.

In this setup, we can lift sections from $H^0(X_1 \cap X_2, E|_{X_1 \cap X_2})$ to $H^0(X, E)$ using the vanishing in item (γ) and the exact sequences

$$\begin{aligned} 0 \rightarrow E(-X_1) \rightarrow E \rightarrow E|_{X_1} \rightarrow 0, \quad \text{and} \\ 0 \rightarrow E|_{X_1}(-X_2) \rightarrow E|_{X_1} \rightarrow E|_{X_1 \cap X_2} \rightarrow 0. \end{aligned}$$

It follows that $F_1^\sharp|_{X''} \cong F_2^\sharp|_{X''}$ if and only if F_1^\sharp and F_2^\sharp are isomorphic on X , cf. [Kob87, Proposition IV.1.7 (2)]. This completes the proof. \square

We formulate now a first separation criterion, describing the geometry of Φ and hence of $M^{\mu ss}$.

Theorem 5.5 (Separating semistable sheaves in the moduli space). *Let F_1 and F_2 be two (H_1, H_2) -semistable sheaves on the projective threefold X such that $F_1^\sharp \not\cong F_2^\sharp$ or $\text{Supp } C_{F_1} \neq \text{Supp } C_{F_2}$. Then, F_1, F_2 give rise to distinct points in $M^{\mu ss}$.*

Proof. We will use the setup and notation given in Section 3.3.1. We look for invariant sections of \mathcal{L}_2^ν on $(R_{red}^{\mu ss})^{wn}$ which separate the orbits corresponding

to F_1 and F_2 . For this we follow the proof of Theorem 3.5 and restrict F_1 and F_2 successively to appropriately chosen general hyperplane sections $X' \in |a_1 H_1|$, $X'' \in |a_2(H_2)_{X'}|$. By [HL10, Cor. 1.1.14] the restrictions of F_1 and F_2 to a general member X' of $|a_1 H_1|$ remain torsion free on X' .

If $F_1^\# \not\cong F_2^\#$ then by Lemma 5.4 their further restrictions to a general $X'' \in |a_2(H_2)_{X'}|$, are locally free, semi-stable and not S -equivalent. Consequently, the corresponding points in the Quot scheme $Q_{X''}$ may be separated by an invariant section σ in a sufficiently high power of \mathcal{L}_0'' . It hence follows from Lemma 3.7 and Remark 3.8 that the map Φ separates the two sheaves F_1 and F_2 .

Suppose now that $F_1^\# \cong F_2^\#$ and that there exists an irreducible component Z of $\text{Supp } C_{F_1}$ which is not contained in $\text{Supp } C_{F_2}$. Then the general element $X' \in |a_1 H_1|$ will cut Z transversally and away from $\text{Supp } C_{F_2}$. We may also assume that X' avoids the singularities of $F_1^\#$. Take $z \in Z \cap X'$. We choose now X'' to go through z but to avoid all other singularities of $F_1|_{X'}$ and of $F_2|_{X'}$. Then on the one hand, $F_2|_{X''}$ is torsion-free and semistable on X'' , as before. On the other hand, we claim that $F_1|_{X''}$ now has torsion in z . Indeed, using the inclusion $F_1 \rightarrow F_1^\#$, the fact that $F_1^\#|_{X'}$ is locally free on X' and that $(F_1^\# / F_1)|_{X'}$ is not trivial in z , one sees that $F_1|_{X'}$ has a singularity at z , hence it is torsion-free but not reflexive at this point. This implies that around z the sheaf $\mathcal{E}xt_{X'}^1(F_1|_{X'}, \omega_{X'})$ is a non-trivial sky-scraper sheaf supported in z ; cf. [HL10, Prop. 1.1.10]. Taking a dévissage of the $(\mathcal{O}_{X',z}/\mathfrak{m}_z^k)$ -module $\mathcal{E}xt_{X'}^1(F_1|_{X'}, \omega_{X'})$ for some large k and making induction on the length we see that no hyperplane section through z on X' can be $\mathcal{E}xt_{X'}^1(F_1|_{X'}, \omega_{X'})$ -regular. In particular, the first morphism of the exact sequence

$$\mathcal{E}xt_{X'}^1(F_1|_{X'}, \omega_{X'}) \rightarrow \mathcal{E}xt_{X'}^1(F_1|_{X'}(-X''), \omega_{X'}) \rightarrow \mathcal{E}xt_{X'}^2(F_1|_{X''}, \omega_{X'})$$

induced by the short exact sequence ($F_1|_{X'}$ is torsion-free)

$$0 \rightarrow F_1|_{X'}(-X'') \rightarrow F_1|_{X'} \rightarrow F_1|_{X''} \rightarrow 0$$

is not injective. This entails $\text{codim}_{X'} \mathcal{E}xt_{X'}^2(F_1|_{X''}, \omega_{X'}) = 2 < 2 + 1$. Therefore, [HL10, Prop. 1.1.10] implies that $F_1|_{X''}$ is not torsion-free.

Summing up, $F_1|_{X''}$ is not torsionfree and hence not semistable, while $F_2|_{X''}$ is semistable. As a consequence, the corresponding orbits in the Quot scheme $Q_{X''}$ are not GIT-semistable and GIT-semistable, respectively; see Lemma 3.7. Hence, there exists an invariant section σ in a sufficiently high power of \mathcal{L}_0'' such that σ vanishes along the orbit corresponding to $F_1|_{X''}$ and is non-zero on the orbit corresponding to $F_2|_{X''}$. Note that the restriction (to X'') of the universal family over $(R_{red}^{\mu ss})^{wn}$ remains flat around the points represented by F_1 and F_2 , since $F_1|_{X'}$ and $F_2|_{X'}$ are torsion free on X' . Thus, the section σ gives rise to a $SL(V)$ -invariant section $l_{\mathcal{F}}(\sigma)$ in some power of \mathcal{L}_2 that vanishes along the orbit corresponding to F_1 and is non-zero at the orbit corresponding

to F_1 ; cf. Remark 3.8. Consequently, the map Φ separates F_1 and F_2 , i.e., these sheaves give rise to different points in the moduli space $M^{\mu ss}$. \square

The following easy example shows that singularities appearing in codimension higher than 2 should have no influence on separation phenomena in $M^{\mu ss}$.

Example 5.6. Let $X = \mathbb{P}^3$ and consider ideal sheaves \mathcal{I}_p of points $p \in X$. These are torsion-free, non-reflexive, and μ -stable. Their double dual is \mathcal{O}_X , thus $\mathcal{I}_p^\# = \mathcal{O}_X$ and $C_{\mathcal{I}_p} = \emptyset$ for all $p \in X$. While trying to imitate the above proof in order to separate \mathcal{I}_p from \mathcal{I}_q for points $p \neq q$, one discovers the following problem: Let X'' be some complete intersection curve passing through p but not through q . Then, the restriction of \mathcal{I}_p to X'' has torsion whereas the restriction of \mathcal{I}_q does not. As before, these restrictions can be separated by some invariant section in some power of \mathcal{L}_0'' . But now, the restricted family is no longer flat at the point represented by \mathcal{I}_p . In order to see this one uses the same type of arguments as in the proof of Theorem 5.5 and checks that $\mathcal{I}_p|_{X'}$ has torsion in p on X' , therefore no hyperplane section through p can be $\mathcal{I}_p|_{X'}$ -regular, which leads to a different Hilbert polynomial for $\mathcal{I}_p|_{X''}$ as opposed to that of $\mathcal{I}_q|_{X''}$. Moreover, there is only one semistable orbit in $Q_{X''}$, the one of $\mathcal{O}_{X''} = \mathcal{I}_q|_{X''}$. Any invariant section has to be constant on this orbit and its "lift" to the open subset of $(R_{red}^{\mu ss})^{wn}$ where the restricted family remains flat will extend as a constant section as well. Thus, \mathcal{I}_q cannot be separated from \mathcal{I}_p in this way.

In fact we may consider $S = X$ as a parameter space for the sheaves \mathcal{I}_p . It coincides with the Gieseker moduli space of these sheaves. Then, a direct computation (cf. [HL10, Ex. 8.1.8]) shows that the determinant bundle \mathcal{L}_2 of the universal family is trivial on S , cf. [HL10, Ex. 8.1.8.ii)]. Thus, distinct points of S cannot be separated by sections of \mathcal{L}_2^ν and the associated moduli space $M^{\mu ss}$ has only one point.

Tensoring for example a null-correlation bundle E with \mathcal{I}_p , $p \in X$, we obtain a positive-dimensional family of higher-rank stable sheaves differing only in codimension three that map to a single point in the corresponding moduli space $M^{\mu ss}$.

5.2. $M^{\mu ss}$ as a compactification of the moduli space of μ -stable reflexive sheaves. By work of Altman and Kleiman [AK80, Thm. 7.4] (in the algebraic category) as well as by Kosarew and Okonek [KO89] (in the analytic category), there exists a (possibly non-separated) coarse moduli space M_{sim} for isomorphism classes of simple coherent sheaves on a fixed projective variety X . Since every μ -stable sheaf is simple, see [HL10, Cor. 1.2.8], it is natural to compare M_{sim} with the newly constructed moduli space $M^{\mu ss}$. In fact, we will show that $M^{\mu ss}$ provides a natural compactification of the moduli space of μ -stable reflexive sheaves (of fixed topological type) on X .

Note that by [BS76, Ch. V, Thm. 2.8] and the characterization of reflexive sheaves given in [HL10, Prop. 1.1.10(4)], reflexivity is an open property in a flat (proper) family. Let $M_{\text{refl}}^{\mu s}$ be the locally closed subscheme of M_{sim} representing (isomorphism classes of) μ -stable reflexive sheaves with class c and determinant Λ . It follows for example from [KO89, Prop. 6.6] and [Kob87, Cor. 7.12] that $(M_{\text{refl}}^{\mu s})_{\text{red}}$ is a separated quasi-projective variety.

Theorem 5.7 (Compactifying the moduli space of stable reflexive sheaves). *There exists a natural morphism*

$$\phi : (M_{\text{refl}}^{\mu s})^{wn} \rightarrow M^{\mu ss}$$

that embeds $(M_{\text{refl}}^{\mu s})^{wn}$ as a Zariski-open subset of $M^{\mu ss}$.

Proof. The proof is divided into five steps.

Step 1: constructing the map ϕ : As in Section 3.3, let $R^{\mu s}$ and $R^{\mu ss}$ denote the locally closed subschemes (of the Quot-scheme used to construct $M^{\mu ss}$) of all μ -stable, or μ -semistable quotients with the chosen invariants, respectively. Moreover, let $R_{\text{refl}}^{\mu s}$ denote the subscheme of reflexive μ -stable quotients. Furthermore, we let $S = (R^{\mu ss})^{wn}$ and $S_{\text{refl}}^{\mu s} = (R_{\text{refl}}^{\mu s})^{wn}$ be the respective weak normalisations, and we note that we have a natural $SL(V)$ -equivariant inclusion

$$(5.1) \quad S_{\text{refl}}^{\mu s} \hookrightarrow S.$$

It follows from [HL10, Lem. 4.3.2] that the center of $SL(V)$ acts trivially on all the spaces introduced above; i.e., the action of $SL(V)$ factors over $PGL(V)$, and moreover that the respective actions of $PGL(V)$ on $R^{\mu s}$ and on $(R^{\mu s})^{wn}$ are set-theoretically free. We will show that the $PGL(V)$ -action is proper on $S_{\text{refl}}^{\mu s}$. For this, it suffices to show that the action on $(S_{\text{refl}}^{\mu s})^{an} =: \mathcal{S}$ is proper in the topological sense. As this action is set-theoretically free, it suffices to show the following two points, cf. [Püt09, Lem. 4]:

- (α) The quotient topology on $\mathcal{S}/PGL(V)$ is Hausdorff, and
- (β) there exists *local slices* through every point of \mathcal{S} ; i.e., through every point $s \in \mathcal{S}$ there exists a locally closed analytic subset $s \in T \subset \mathcal{S}$ such that $PGL(V) \cdot T$ is open in \mathcal{S} and such that the map $PGL(V) \times T \rightarrow PGL(V) \cdot T \subset \mathcal{S}$ is biholomorphic.

If we set $\mathcal{M} := ((M_{\text{refl}}^{\mu s})^{wn})^{an}$, then \mathcal{M} is the (analytic) coarse moduli space for families of μ -stable reflexive sheaves with the chosen invariants parametrised by weakly normal complex base spaces. Consequently, the restriction of the universal family from R to \mathcal{S} gives rise to a holomorphic classifying map $\pi : \mathcal{S} \rightarrow \mathcal{M}$. Since isomorphism classes of sheaves parametrised by \mathcal{S} are

realised by the $PGL(V)$ -action, cf. [HL10, Sect. 4.3], the map π induces an injective continuous map $\mathcal{S}/PGL(V) \rightarrow \mathcal{M}$. Since \mathcal{M} is Hausdorff, $\mathcal{S}/PGL(V)$ is likewise Hausdorff. This shows (α) .

Let now $s_0 \in \mathcal{S}$, and $\pi(s_0)$ the corresponding point in the moduli space \mathcal{M} . Then, we may find an open neighbourhood W of $\pi(s_0)$ in \mathcal{M} such that there exists a universal family \mathcal{U} over $W \times X$, see [KO89, Thm. 6.4] or [AK80, Thm. 7.4]. After shrinking W is necessary, the family \mathcal{U} induces a holomorphic section $\sigma : W \rightarrow \pi^{-1}(W) \subset \mathcal{S}$ of $\pi|_{\pi^{-1}(W)}$ through $s_0 \in \mathcal{S}$. Since every fibre of π is a $PGL(V)$ -orbit, we conclude that $PGL(V) \cdot \sigma(W) = \pi^{-1}(W)$ is open in \mathcal{S} . Moreover, since $\pi|_{\sigma(W)} : \sigma(W) \rightarrow W$ is biholomorphic, hence bijective, and since \mathcal{M} parametrises isomorphism classes of μ -stable reflexive sheaves, for any $s \in \sigma(W) =: T$ we have

$$PGL(V) \cdot s \cap \sigma(W) = \{s\}.$$

As a consequence, the natural map $\eta : PGL(V) \times T \rightarrow PGL(V) \cdot T$ is holomorphic, open, and bijective. As $PGL(V) \cdot T \subset \mathcal{S}$ is weakly normal, η is therefore biholomorphic; i.e., $T = \sigma(W)$ is a local holomorphic slice through s_0 .

To sum up, we have established that $PGL(V)$ acts properly on $S_{\text{refl}}^{\mu s}$. As a consequence, the geometric quotient $S_{\text{refl}}^{\mu s}/PGL(V)$ exists in the category of algebraic spaces [Kol97], and once the existence of this quotient is established, it is rather straightforward to see that in fact $(M_{\text{refl}}^{\mu s})^{wn} \cong S_{\text{refl}}^{\mu s}/PGL(V)$. By abuse of notation, we will denote the corresponding quotient map $S_{\text{refl}}^{\mu s} \rightarrow (M_{\text{refl}}^{\mu s})^{wn}$ by π .

Recalling the inclusion (5.1), we may restrict the $SL(V)$ -invariant and hence $PGL(V)$ -invariant map Φ (cf. Definition 4.4) to $S_{\text{refl}}^{\mu s}$. As π is a categorical quotient, the resulting map descends to a regular morphism $\phi : (M_{\text{refl}}^{\mu s})^{wn} \rightarrow M^{\mu ss}$ completing the following diagram:

$$(5.2) \quad \begin{array}{ccc} S_{\text{refl}}^{\mu s} & \xhookrightarrow{\quad} & S \\ \pi \downarrow & & \downarrow \Phi \\ (M_{\text{refl}}^{\mu s})^{wn} & \xrightarrow{\quad \phi \quad} & M^{\mu ss}. \end{array}$$

This concludes the construction of the desired map from $(M_{\text{refl}}^{\mu s})^{wn}$ to $M^{\mu ss}$.

Step 2: ϕ is injective: This follows immediatly from Lemma 5.4.

Step 3: $\phi((M_{\text{refl}}^{\mu s})^{wn})$ is open: The set $A := S \setminus S_{\text{refl}}^{\mu s}$ is a $SL(V)$ -invariant subvariety of S . It follows from Proposition 4.2 that its image $\Phi(A) \subset M^{\mu ss}$ is closed. Furthermore, as a consequence of Lemma 5.4 we deduce that A is Φ -saturated, i.e., $\Phi^{-1}(\Phi(A)) = A$. Consequently, the set

$$U := \phi((M_{\text{refl}}^{\mu s})^{wn}) = \Phi(S_{\text{refl}}^{\mu s}) = M^{\mu ss} \setminus \Phi(A)$$

is open, as claimed.

Step 4: ϕ is open as a map onto its image U : Since we have already seen that $\phi : (M_{\text{refl}}^{\mu s})^{wn} \rightarrow U$ is bijective, it suffices to show that $\phi : (M_{\text{refl}}^{\mu s})^{wn} \rightarrow U$ is closed. Let \hat{Z} be any closed subvariety of $(M_{\text{refl}}^{\mu s})^{wn}$. Then, let $Z \subset S_{\text{refl}}^{\mu s}$ be its preimage under π , and \bar{Z} the closure of Z in S , which is automatically $SL(V)$ -invariant. As a consequence of Proposition 4.2, the image $\Phi(\bar{Z})$ is closed in $M^{\mu ss}$. Hence, $\phi(\hat{Z}) = \Phi(\bar{Z}) \cap U$ is closed in U , as claimed.

Step 5: conclusion of proof: Summarising the previous steps, we know that $\phi : (M_{\text{refl}}^{\mu s})^{wn} \rightarrow U$ is a bijective open morphism. Hence, its (set-theoretical) inverse ϕ^{-1} is continuous. Since $U \subset M^{\mu s}$ is weakly normal by construction, it follows that ϕ^{-1} is regular, and hence that ϕ is an isomorphism, as claimed. \square

As a corollary of the proof of Theorem 5.7 we obtain the following slightly finer result:

Corollary 5.8 (The equivariant geometry of $S_{\text{refl}}^{\mu ss}$). *If we denote by $S_{\text{refl}}^{\mu s}$ the open $SL(V)$ -invariant subvariety of S that parametrises reflexive μ -stable sheaves and by $\partial S_{\text{refl}}^{\mu s} = S \setminus S_{\text{refl}}^{\mu s}$ its complement, the following holds:*

- (i) *The geometric quotient of $S_{\text{refl}}^{\mu s}$ by $SL(V)$ exists. It is isomorphic to $(M_{\text{refl}}^{\mu s})^{wn}$.*
- (ii) *The map $S_{\text{refl}}^{\mu s}/SL(V) \rightarrow M^{\mu ss}$ induced by the $SL(V)$ -invariant morphism $\Phi|_{S_{\text{refl}}^{\mu s}}$ is an open embedding.*
- (iii) *The image of $\partial S_{\text{refl}}^{\mu s}$ under Φ is equal to the complement of $S_{\text{refl}}^{\mu s}/SL(V)$ in $M^{\mu ss}$; in other words, $S_{\text{refl}}^{\mu s}$ is Φ -saturated in S .*

Remark 5.9. Theorem 5.7 above together with Proposition 6.7 below solves an old problem (raised for example independently by Tyurin and Teleman) of exhibiting a sheaf-theoretically and geometrically meaningful compactification of the gauge-theoretic moduli space of vector bundles that are stable with respect to a chosen Kähler class $[\omega]$ on a given Kähler manifold, in the particular case of projective manifolds and classes $[\omega] \in \text{Amp}(X)_{\mathbb{R}}$. In particular, $M^{\mu ss}$ fulfills the properties demanded of a “nice” compactification in [Tel08, Sect. 3.2, Conj. 1].

6. APPLICATION TO WALL-CROSSING PROBLEMS

6.1. Motivation – Qin’s and Schmitt’s work. Investigating wall crossing phenomena for moduli spaces of Gieseker semistable sheaves over higher dimensional bases, Qin adapts his notion of “wall” from the two-dimensional to the higher-dimensional case. However, in contrast to the surface case, he immediately finds examples of varieties (with Picard number three) where these walls are not locally finite inside the ample cone, see [Qin93, Ex. I.2.3].

In order to avoid these pathologies, Schmitt [Sch00] restricts to segment between integral ample classes and is able to prove statements similar to the case of a two dimensional base [MW97] provided that wall crossing occurs on a rational wall. However, he also gives examples of threefolds where this condition is not satisfied. More precisely, there exist threefolds X with Picard number equal to two and rank two vector bundles E on X which are μ -stable with respect to some integral ample divisor H_0 and unstable with respect to some other integral ample divisor H_1 , and moreover such that the class $H_\lambda := (1 - \lambda)H_0 + \lambda H_1$ for which E becomes strictly semistable is irrational, see [Sch00, Ex. 1.1.5]. This irrationality is due to observation that $\lambda \in \mathbb{R}$ is obtained as a solution of a quadratic equation given by the condition $H_\lambda^2 D = 0$ for a suitable rational divisor D , cf. Section 6.4.

In the subsequent sections, we will solve these problems based on the philosophy that the natural "polarisations" to consider when defining slope-semistability on higher dimensional base manifolds are not ample divisors but rather movable curves, cf. [Miy87, CP11]. More precisely, given an n -dimensional smooth projective variety X , we consider the open set $P(X) \subset H_{\mathbb{R}}^{1,1}(X)$ of powers $[H]^{n-1}$ of ample divisor classes $[H] \in \text{Amp}(X)$ inside the cone of movable curves. We show that $P(X)$ is open and that the natural map $\text{Amp}(X) \rightarrow P(X)$ (taking $n - 1$ -st powers) is an isomorphism, see Proposition 6.5. Moreover, we show that $P(X)$ supports a locally finite chamber structure given by linear rational walls such that the notion of slope-semistability is constant within each chamber, see Theorem 6.6. Moreover, any chamber (even if it is not open) contains products $H_1 H_2 \dots H_{n-1}$ of integral ample divisor classes, see Proposition 6.7.

6.2. Boundedness. As discussed in Section 2.2, when one is concerned with the variation of slope-stability of torsion-free sheaves on a projective manifold X of dimension $n > 2$ the relevant space of polarizations to look at should be that of classes of positive curves rather than positive divisors, see also [Tom10].

Recalling some notions already introduced in Section 2.2, let $N_1 = N_1(X)_{\mathbb{R}}$ be the space of 1-cycles modulo numerical equivalence on X with real coefficients, and let $N^1 = N^1(X)_{\mathbb{R}}$ the dual space of divisor classes containing the open cone $\text{Amp}(X)$ of real ample divisor classes. Consider the associated subset $P(X)$ of N_1 consisting of $(n - 1)$ -st powers of real ample classes in N^1 . For the purposes of this section we shall only need polarisations from $P(X)$. When H is an ample class we shall speak of *(semi)stability with respect to H* meaning (semi)stability with respect to the complete intersection class H^{n-1} .

Since there is no added complication, in the following preparatory statements we consider slope-semistability with respect to arbitrary Kähler classes. Again we say that a torsion-free sheaf is (semi)stable with respect to some

Kähler class ω if it is (semi)stable with respect to ω^{n-1} . We first need to adapt a preliminary boundedness result from [Tel08] to our situation.

Lemma 6.1. *Let X be a compact complex manifold of dimension n endowed with a Kähler class ϕ and E a reflexive sheaf on X . Then for any $d \in \mathbb{R}$ the set $BN(E)_{\geq d} := \{L \in \text{Pic}(X) \mid \text{Hom}(L, E) \neq 0, \deg_{\phi} L \geq d\}$ is compact in $\text{Pic}(X)$.*

Proof. By the flattening result of Raynaud [Ray72] and Hironaka's theorem on elimination of points of indeterminacy [Hir64], there is a composition of blow-ups of smooth centers $f : X' \rightarrow X$ such that $E' := f^*E/\mathcal{T}or(f^*E)$ is locally free. Denote by E_1, \dots, E_k the irreducible components of the exceptional divisor of f in X' . Then $\phi' := f^*\phi - a_1[E_1] - a_2[E_2] - \dots - a_k[E_k] \in N^1(X')_{\mathbb{R}}$ is a Kähler class on X' for suitably chosen (small) positive real numbers a_i , and we shall compute degrees on X' with respect to this class. Now if L belongs to $BN(E)_{\geq d}$ then $f^*L \in BN(E')_{\geq d}$. Rephrasing, we have just shown that the image of $BN(E)_{\geq d}$ under the injective holomorphic map $f^* : \text{Pic}(X) \rightarrow \text{Pic}(X')$ is contained in $BN(E')_{\geq d}$. This latter set is compact by [Tel08, Prop. 2.5], hence bounded, and consequently $BN(E)_{\geq d}$ is likewise bounded. Using Grauert's Semicontinuity Theorem one sees that $BN(E)_{\geq d}$ is closed in $\text{Pic}(X)$, and therefore compact, as X is Kähler. This concludes the proof. \square

Lemma 6.2. *Let X be a compact complex manifold of dimension n and let $\phi_0, \phi_1 \in H^{1,1}(X, \mathbb{R})$ be two Kähler classes on X . For every $\tau \in [0, 1]$ we set*

$$\phi_{\tau} := (1 - \tau)\phi_0 + \tau\phi_1 \in H^{1,1}(X, \mathbb{R}).$$

Suppose that the torsion-free sheaf E on X is semistable with respect to ϕ_1 and unstable with respect to ϕ_0 , and let

$$t := \inf\{\tau > 0 \mid E \text{ is semistable with respect to } \phi_{\tau}\}.$$

Then, E is properly semistable with respect to ϕ_t .

Proof. The continuity of $\tau \mapsto \deg_{\phi_{\tau}}(E)$ ([Miy87, Cor. 2.4]) implies that E is ϕ_t -semi-stable. We shall show that it is not ϕ_t -stable. By the definition of t there exists an integer k with $0 < k < \text{rk } E$ and an increasing sequence $(\tau_n)_{n \in \mathbb{N}}$ of real numbers converging to t together with a corresponding sequence (F_n) of ϕ_{τ_n} -destabilising rank k subsheaves of E . Then, for each $n \in \mathbb{N}$, the reflexive exterior power $(\bigwedge^k F_n)^{\vee\vee}$ is an invertible subsheaf of $(\bigwedge^k E)^{\vee\vee}$ that is destabilising with respect to ϕ_{τ_n} . By Lemma 6.1 applied to $\phi = \phi_t$ and $d = \mu_{\phi_t}(\bigwedge^k F_n)^{\vee\vee} - 1$, and by our assumption on t we get an invertible subsheaf L of $(\bigwedge^k E)^{\vee\vee}$ whose slope with respect to ϕ_t equals the slope of $(\bigwedge^k E)^{\vee\vee}$ by continuity of the degree. Moreover, by taking saturation if necessary, we may assume that L is a saturated subsheaf of $(\bigwedge^k E)^{\vee\vee}$. We will show that $L = (\bigwedge^k F)^{\vee\vee}$ for some subsheaf F of E , which will have the same slope

with respect to ϕ_t as E and thus prove its proper ϕ_t -semistability. For this we follow some ideas contained in [Tel08, Sect. 2.2]. There is no restriction of generality in supposing that E is reflexive, which implies that there exists a Zariski-open subset U of X with $\text{codim}_X(X \setminus U) \geq 2$ such that $E|_U$ is locally free. Moreover, since L is saturated in E , there exists an open subset $U' \subset U$ with $\text{codim}_X(X \setminus U') \geq 2$ such that $L|_{U'}$ is a line subbundle of $E|_{U'}$. Let $C_k(E|_{U'}) \subset \bigwedge^k E_{U'}$ be the cone subbundle over the relative Grassmannian $G_s(E|_{U'})$. Then, over U' the line bundle L is contained in $C_k(E)$ and thus gives rise to a subbundle F' of $E|_{U'}$ via projection to the relative Grassmannian, cf. [Tel08, first paragr. of Sect. 2.2]. If F is the unique extension of F' as a reflexive subsheaf of E , then we have $L = \bigwedge^k(F)^{\vee\vee}$, as desired. Moreover, by continuity we have $\mu_{\phi_t}(F) \geq \mu_{\phi_t}(E)$. Since E is ϕ_t -semistable, it has to be properly ϕ_t -semistable. \square

The next proposition proves boundedness of semistable torsion-free sheaves with respect to Kähler classes. Although we do not need the result in this generality, the techniques involved will be needed in the special case of polarizations from $P(X)$.

Proposition 6.3 (Boundedness). *Let X be a projective manifold of dimension n , and let K a compact subset of the Kähler cone $\mathcal{K}(X) \subset H^{1,1}(X, \mathbb{R})$ of X . Fix a natural number $r > 0$ and classes $c_i \in H^{2i}(X, \mathbb{R})$. Then, the family of rank r torsion-free sheaves E with $c_i(E) = c_i$ that are semistable with respect to some polarization from K is bounded.*

Proof. Let ϕ_1 be some element of K . Choose an ample class ϕ_0 in $H^{1,1}(X, \mathbb{R})$. For $\tau \in [0, 1]$ set

$$\phi_\tau := (1 - \tau)\phi_0 + \tau\phi_1,$$

and denote by μ_τ the slope with respect to ϕ_τ . We shall prove boundedness by applying [HL10, Thm. 3.3.7] to our family with respect to the ample polarization ϕ_0 . In order to establish the assertion of Proposition 6.3, it will thus suffice to prove the following.

Lemma 6.4. *In the setup of Proposition 6.3, if E_{\max} is the maximally ϕ_0 -destabilising subsheaf of a ϕ_1 -semistable torsion-free sheaf E of rank r , then $\mu_0(E_{\max})$ is bounded by a constant depending only on $c_1(E)$, $c_2(E)$, ϕ_0 and K .*

Proof. The idea of the proof is to produce a filtration of E whose graduation has ϕ_0 -semi-stable terms whose slope with respect to ϕ_0 is bounded by a constant depending only on $c_1(E)$, $c_2(E)$, ϕ_0 and K . This constant will then bound $\mu_0(E_{\max})$ as well.

Let $E_0 = E$ be ϕ_1 -semistable and set

$$t_1 := \inf\{\tau > 0 \mid E_0 \text{ is semistable with respect to } \phi_\tau\}.$$

If $t_1 = 0$, then E is ϕ_0 -semi-stable and the claim is verified. We therefore assume in the following that $t_1 > 0$. Under this assumption, we know by Lemma 6.2 that E_0 can be written as an extension

$$(6.1) \quad 0 \rightarrow E_1 \rightarrow E_0 \rightarrow E_2 \rightarrow 0$$

with a torsion-free subsheaf E_1 and torsion-free quotient E_2 , such that E_1 and E_2 are both ϕ_{t_1} -semi-stable with slopes

$$(6.2) \quad \mu_{t_1}(E_1) = \mu_{t_1}(E_2) = \mu_{t_1}(E_0).$$

We define

$$(6.3) \quad a_1 := \frac{\mathrm{rk}(E_2)c_1(E_1) - \mathrm{rk}(E_1)c_1(E_2)}{\mathrm{rk} E_0}$$

and we note that (6.1) as well as additivity of Chern classes in short exact sequences leads to

$$(6.4) \quad a_1 = \frac{\mathrm{rk}(E_2)c_1(E_0)}{\mathrm{rk} E_0} - c_1(E_2) = c_1(E_1) - \frac{\mathrm{rk}(E_1)c_1(E_0)}{\mathrm{rk} E_0}.$$

As a consequence of (6.2), we hence obtain

$$(6.5) \quad a_1 \phi_{t_1}^{n-1} = 0,$$

i.e., a_1 is ϕ_{t_1} -primitive. The Hodge Index Theorem [Voi02, Thm. 6.32] implies that

$$(6.6) \quad a \mapsto -a^2 \phi_{t_1}^{n-2}$$

defines the square of a norm on $\mathrm{Ker}(\phi_{t_1}^{n-1}) \subset NS(X)_{\mathbb{R}}$. In particular, using (6.5) this implies

$$(6.7) \quad 0 \leq -a^2 \phi_{t_1}^{n-2},$$

with equality if and only if $a_1 = 0$.

We show now that $-a_1^2 \phi_{t_1}^{n-2}$ is bounded from above by some constant that only depends on $c_1(E)$, $c_2(E)$, ϕ_1 and K . This will imply that a_1 is contained in a finite set that depends only on $c_1(E)$, $c_2(E)$, ϕ_0 and K and (6.4) will give a bound on $c_1(E_1)$, and consequently also on $c_1(E_2)$ (note that ϕ_{t_1} belongs to the convex hull of K and ϕ_0).

Before we proceed, recall the definition of the discriminant of a torsion-free sheaf F , cf. [HL10, Sect. 3.4]:

$$\Delta(F) = \frac{1}{\mathrm{rk} F} \left(c_2(F) - \frac{\mathrm{rk}(F) - 1}{2 \mathrm{rk} F} c_1^2(F) \right).$$

A short computation shows that we can express the discriminant of E_0 in terms of a_1 and the discriminants of E_1 and E_2 , as follows:

$$(6.8) \quad \Delta(E_0) = -\frac{1}{2 \mathrm{rk}(E_1) \mathrm{rk}(E_2)} a_1^2 + \frac{\mathrm{rk} E_1}{\mathrm{rk} E_0} \Delta(E_1) + \frac{\mathrm{rk} E_2}{\mathrm{rk} E_0} \Delta(E_2).$$

Since E_1 and E_2 are both ϕ_{t_1} -semi-stable, the Bogomolov inequality (see [BS94, Cor. 3] for the case of polystable reflexive sheaves and [BM10, Lem. 2.1] for the general case) holds for both sheaves; i.e., we have

$$(6.9) \quad \Delta(E_i)\phi_{t_1}^{n-2} \geq 0 \quad \text{for } i = 1, 2.$$

Combining the lower bound (6.7) with the expression (6.8) and the Bogomolov inequalities (6.9) we infer that

$$0 \leq -\frac{1}{2\text{rank}(E_1)\text{rank}(E_2)}a_1^2\phi_{t_1}^{n-2} \leq \Delta(E_0)\phi_{t_1}^{n-2},$$

which establishes the desired boundedness for a_1 .

We now iterate this argument. For this, we set

$$t_2 := \inf\{\tau > 0 \mid E_1, E_2 \text{ are semi-stable with respect to } \phi_\tau\}.$$

If $t_2 = 0$ we are done as before, for $0 \subset E_1 \subset E$ is the desired filtration.

When $t_2 \neq 0$ one of E_1, E_2 will be properly ϕ_{t_2} -semi-stable and the other ϕ_{t_2} -semi-stable. For simplicity of notation suppose that E_2 is properly ϕ_{t_2} -semi-stable and denote by E_3 a subsheaf of E_2 with torsion-free quotient E_4 such that E_3 and E_4 are ϕ_{t_2} -semi-stable and $\mu_{t_2}(E_3) = \mu_{t_2}(E_4) = \mu_{t_2}(E_2)$. In analogy with the definition of a_1 , cf. (6.3), we set

$$\begin{aligned} a_2 &:= \frac{\text{rank}(E_4)c_1(E_3) - \text{rank}(E_3)c_1(E_4)}{\text{rank}E_2} \\ &= \frac{\text{rank}(E_4)c_1(E_2)}{\text{rank}E_2} - c_1(E_4) = c_1(E_3) - \frac{\text{rank}(E_3)c_1(E_2)}{\text{rank}E_2}. \end{aligned}$$

As in the first step, we see that a_2 is ϕ_{t_2} -primitive. Furthermore, comparing discriminants we arrive at

$$\begin{aligned} \Delta(E_0) + \frac{1}{2\text{rank}(E_1)\text{rank}(E_2)}a_1^2 &= \frac{\text{rank}E_1}{\text{rank}E_0}\Delta(E_1) + \frac{\text{rank}E_2}{\text{rank}E_0}\Delta(E_2) \\ &= \frac{\text{rank}E_1}{\text{rank}E_0}\Delta(E_1) + \frac{\text{rank}E_2}{\text{rank}E_0} \times \\ &\quad \left(-\frac{1}{2\text{rank}(E_3)\text{rank}(E_4)}a_2^2 + \frac{\text{rank}E_3}{\text{rank}E_2}\Delta(E_3) + \frac{\text{rank}E_4}{\text{rank}E_2}\Delta(E_4) \right) \\ &= -\frac{\text{rank}E_2}{2\text{rank}(E_0)\text{rank}(E_3)\text{rank}(E_4)}a_2^2 \\ &\quad + \frac{\text{rank}E_1}{\text{rank}E_0}\Delta(E_1) + \frac{\text{rank}E_3}{\text{rank}E_0}\Delta(E_3) + \frac{\text{rank}E_4}{\text{rank}E_0}\Delta(E_4). \end{aligned}$$

As above, the Hodge Index Theorem and the Bogomolov inequality then imply boundedness of $a_2, c_1(E_3), c_1(E_4)$ by some function depending only on $c_1(E), c_2(E), \phi_0$ and K .

Since rank 1 torsion-free sheaves are semi-stable with respect to any polarization, the process stops after at most $r - 1$ steps. It produces a filtration of E such that the associated graduation has ϕ_0 -semi-stable torsion-free terms whose slopes with respect to ϕ_0 are bounded by some constant $C = C(c_1(E), c_2(E), \phi_0, K)$ depending only on $c_1(E)$, $c_2(E)$, ϕ_0 , K .

Finally, the inclusion $E_{\max} \subset E$ gives a nontrivial morphism from E_{\max} to some term of this graduation showing that $\mu_0(E_{\max}) \leq C$. This concludes the proof of Lemma 6.4. \square

As already noted above, Lemma 6.4 implies Proposition 6.3 by [HL10, Prop. 3.3.7]. This concludes the proof of Proposition 6.3. \square

6.3. A chamber structure on the set of $(n - 1)^{\text{st}}$ powers of ample classes. We now return to our original setup and deal with polarisation from $P(X)$. We first note the following basic property.

Proposition 6.5 (Injectivity of power maps). *The set $P(X)$ is open in N_1 , and the map $\alpha \mapsto \alpha^{n-1}$ is a bijection from $\text{Amp}(X)$ to $P(X)$.*

Proof. We put norms $\|\cdot\|_k$ on the real vector spaces $H_{\mathbb{R}}^{k,k}(X)$. For $1 \leq k \leq n$ the continuity of the maps $p_k : H_{\mathbb{R}}^{1,1}(X) \rightarrow H_{\mathbb{R}}^{k,k}(X)$, $\alpha \mapsto \alpha^k$ implies the existence of constants C_k such that $\|\alpha^k\|_k \leq C_k \|\alpha\|_1^k$ and further that the total derivative of p_{n-1} at α is the map $\beta \mapsto (n-1)\alpha^{n-2} \cdot \beta$. Thus the restriction of p_{n-1} to the ample cone is a local isomorphism by Hard Lefschetz [Voi02, Thm. 6.25], and consequently, the image $P(X)$ of p_{n-1} is open.

Let α, β be two real ample classes such that $\alpha^{n-1} = \beta^{n-1}$ in N_1 . Multiplication by α from the left and by β from the right gives

$$(6.10) \quad \alpha^n = \alpha\beta^{n-1} \quad \text{as well as} \quad \alpha^{n-1}\beta = \beta^n,$$

and hence

$$(6.11) \quad \alpha^n \beta^n = (\alpha^{n-1}\beta)(\alpha\beta^{n-1}).$$

On the other hand, the Khovanskii-Teissier inequalities [Laz04, Ex. 1.6.4] give

$$(6.12) \quad (\alpha^{n-j}\beta^j)(\alpha^{n-j-2}\beta^{j+2}) \leq (\alpha^{n-j-1}\beta^{j+1})^2 \quad \text{for } 0 \leq j \leq n-2.$$

Multiplying all of these inequalities, we obtain the inequality

$$(6.13) \quad \alpha^n \beta^n \leq (\alpha^{n-1}\beta)(\alpha\beta^{n-1}).$$

Note that in our setup (6.11) says that equality is attained in (6.13). Thus equality must hold in each of the Khovanskii-Teissier inequalities (6.12) above. Together with the equalities (6.10), this immediately implies that all the mixed intersection products $\alpha^{n-j}\beta^j$, $0 \leq j \leq n$, are equal. It follows that

$$(\alpha - \beta)\alpha^{n-1} = 0;$$

i.e., $\alpha - \beta$ is a primitive cohomology class with respect to the polarisation α . Recall that the Hodge Index Theorem [Voi02, Thm. 6.32] states that the quadratic form $q(\gamma) := \gamma^2 \alpha^{n-2}$ is definite on the primitive part of $H_{\mathbb{R}}^{1,1}(X)$. Therefore, setting $\gamma = \alpha - \beta$ and invoking again the equality of mixed intersection products, we conclude that $\alpha = \beta$, which was to be shown. \square

As in the 2-dimensional case, we obtain a locally finite linear rational chamber decomposition, this time however not on the ample cone, but on $P(X) \subset N_1$.

Theorem 6.6 (Chamber structure on $P(X)$). *For any set of topological invariants (r, c_1, \dots, c_n) of torsion-free sheaves on X , there is a locally finite system of linear rational walls on $P(X)$ defining a chamber structure with the following property: if two elements α and β in $P(X)$ belong to the same chamber then for any torsion free coherent sheaf F with the given topological invariants, F is α -(semi-)stable if and only if F is β -(semi-)stable.*

Proof. Let α be any real ample class in N^1 and K some compact neighbourhood of α^{n-1} in $P(X)$. If some element $\phi^{n-1} \in K$ belongs to a wall, by Lemma 6.2 there exists some properly ϕ -semistable sheaf E on X . If E' is some subsheaf of rank r' of E contradicting the ϕ -stability of E we have $(rc_1(E') - r'c_1(E)) \cdot \phi^{n-1} = 0$. Thus ϕ^{n-1} belongs to the rational hyperplane $(rc_1(E') - r'c_1(E))^{\perp}$ in N_1 . It suffices to show that the set of classes of the type $(rc_1(E') - r'c_1(E))$ for E and E' as above is finite. But this follows from the same argument we used in the proof of Lemma 6.4. \square

6.4. Explaining the pathologies found by Schmitt and Qin. By Proposition 6.5 our chamber structure on $P(X)$ pulls back to a locally finite chamber structure on $\text{Amp}(X)$. The corresponding walls thus obtained in $\text{Amp}(X)$ are given by equations that are homogeneous of degree $n - 1$, so, except in the case when $\rho(X) := \dim N^1(X) \leq 2$, these need not be linear. This explains the pathologies encountered in the approaches of Schmitt and Qin.

More precisely, on the one hand Schmitt [Sch00] considers segments connecting rational points in $\text{Amp}(X)$ as well as points where the induced notion of slope-stability changes. These separating points are precisely the intersection points of his segments with our walls. This clarifies the appearance of non-rational points as for example in [Sch00, Ex. 1.1.5]. On the other hand, these intersection points are also contained in the linear walls considered by Qin in [Qin93]. This in turn explains the pathologies of Qin's linear chamber structure on $\text{Amp}(X)$, and in particular the fact that it cannot be locally finite in general.

6.5. Representing chambers by complete intersection curves. The wall system we defined gives an obvious stratification of $P(X)$ into connected chambers; also chambers which are not of the maximal dimension $\rho(X)$ are considered here. We show next that every such chamber contains a class which is an intersection of integral ample divisor classes, cf. the discussion in Section 6.1.

Proposition 6.7 (Representing chambers by complete intersections). *Let X be a projective manifold of dimension $n > 2$ and fix some chamber $\mathcal{C} \subset P(X)$ of stability polarizations in $P(X)$. Then, there exist some ample integral classes A and B such that the complete intersection class $A^{n-2}B$ lies in \mathcal{C} .*

Proof. For any $H \in \text{Amp}(X)$, the \mathbb{R} -linear map $L_H \in L(N^1, N_1)$ given by $L_H(D) := DH^{n-2}$ is invertible by Hard Lefschetz. As the map $\text{Amp}(X) \rightarrow L(N_1, N^1)$, $H \mapsto L_H^{-1}$ is continuous, the same also holds for the map

$$e : \text{Amp}(X) \times N_1 \rightarrow N^1, \quad (H, C) \mapsto L_H^{-1}(C).$$

Take $H \in \text{Amp}(X)$ such that H^{n-1} lies in the fixed chamber $\mathcal{C} \subset P(X)$. We have $e(H, H^{n-1}) = H$. Since the chambers are cut out by rational walls by Theorem 6.6, close to H^{n-1} there exists a rational element C in the same chamber as H^{n-1} . Furthermore, choose a rational ample class $A \in \text{Amp}(X)$ close to H . Then, $B := e(A, C)$ is close to H , and hence in particular, B is in $\text{Amp}(X)$. By construction, we have

$$(6.14) \quad C = BA^{n-2}.$$

Since C and A are rational, we infer that the intersection numbers $BA^{n-2}D = CD$ are rational whenever $D \in N^1$ is rational. But the elements $A^{n-2}D$ span $N_1(X)_{\mathbb{Q}}$ as D runs through $N^1(X)_{\mathbb{Q}}$. Hence, B is a rational element in $\text{Amp}(X)$. Together with equation (6.14) and with the observation that taking positive real scalar multiples in $\text{Amp}(X)$ or $P(X)$ does not change the induced notion of slope-(semi)stability, this implies the claim. \square

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